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ABSTRACT

This is one of a series that is a collection of translations from the extensive Soviet literature of the past 25 years on research in the psychology of mathematics instruction. It also includes works on methods of teaching mathematics directly influenced by the psychological research. Selected papers and books considered to be of value to the American mathematics educator have been translated from the Russian and appear in this series for the first time in English. The aim of this series is to acquaint mathematics educators and teachers with directions, ideas, and accomplishments in the psychology of mathematical instruction in the Soviet Union. The authors have done a philosophical and psychological analysis of the special problems of teaching mathematics to children. They first create a psychological construct, then develop techniques that follow. The six chapter titles are: The Psychological and Didactic Principles of Teaching Arithmetic: The Introduction of Numbers, Counting, and the Arithmetical Operations: Instruction in Mental and Written Calculation: Teaching Problem Solving: Geometry in the Primary Grades: and Different Kinds of Pupils and How to Approach Them in Arithmetic Instruction. (MK)

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SOVIET STUDIES IN THE PSYCHOLOGY OF LEARNING AND TEACHING MATHEMATICS

VOLUME XIV

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EDUCATION

Mary L. Charles
NSF

IE 030 458

SCHOOL MATHEMATICS STUDY GROUP
STANFORD UNIVERSITY
AND
SURVEY OF RECENT EAST EUROPEAN
MATHEMATICAL LITERATURE
THE UNIVERSITY OF CHICAGO

SOVIET STUDIES

IN THE

PSYCHOLOGY OF LEARNING

AND TEACHING MATHEMATICS

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VOLUME XIV

TEACHING ARITHMETIC IN THE ELEMENTARY SCHOOL

VOLUME EDITOR

JOSEPH R. HOOTEN

The University of Georgia

SCHOOL MATHEMATICS STUDY GROUP
STANFORD UNIVERSITY
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PREFACE

The series Soviet Studies in the Psychology of Learning and Teaching Mathematics is a collection of translations from the extensive Soviet literature of the past twenty-five years on research in the psychology of mathematical instruction. It also includes works on methods of teaching mathematics directly influenced by the psychological research. The series is the result of a joint effort by the School Mathematics Study Group at Stanford University, the Department of Mathematics Education at the University of Georgia, and the Survey of Recent East European Mathematical Literature at the University of Chicago. Selected papers and books considered to be of value to the American mathematics educator have been translated from the Russian and appear in this series for the first time in English.

Research achievements in psychology in the United States are outstanding indeed. Educational psychology, however, occupies only a small fraction of the field, and until recently little attention has been given to research in the psychology of learning and teaching particular school subjects.

The situation has been quite different in the Soviet Union. In view of the reigning social and political doctrines, several branches of psychology that are highly developed in the U.S. have scarcely been investigated in the Soviet Union. On the other hand, because of the Soviet emphasis on education and its function in the state, research in educational psychology has been given considerable moral and financial support. Consequently, it has attracted many creative and talented scholars whose contributions have been remarkable.*

Even prior to World War II, the Russians had made great strides in educational psychology. The creation in 1943 of the Academy of Pedagogical Sciences helped to intensify the research efforts and programs in this field. Since then the Academy has become the chief educational research and development center for the Soviet Union. One of the main aims of the Academy is to conduct research and to train research scholars

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*A study indicates that 37.5% of all materials in Soviet psychology published in one year was devoted to education and child psychology. See Contemporary Soviet Psychology by Josef Brozek (Chapter 7 of Present-Day Russian Psychology, Pergamon Press, 1966).

in general and specialized education, in educational psychology, and in methods of teaching various school subjects.

The Academy of Pedagogical Sciences of the USSR comprises ten research institutes in Moscow and Leningrad. Many of the studies reported in this series were conducted at the Academy's Institute of General and Polytechnical Education, Institute of Psychology, and Institute of Defectology, the last of which is concerned with the special psychology and educational techniques for handicapped children.

The Academy of Pedagogical Sciences has 31 members and 64 associate members, chosen from among distinguished Soviet scholars, scientists, and educators. Its permanent staff includes more than 650 research associates, who receive advice and cooperation from an additional 1,000 scholars and teachers. The research institutes of the Academy have available 100 "base" or laboratory schools and many other schools in which experiments are conducted. Developments in foreign countries are closely followed by the Bureau for the Study of Foreign Educational Experience and Information.

The Academy has its own publishing house, which issues hundreds of books each year and publishes the collections Izvestiya Akademii Pedagogicheskikh Nauk RSFSR [Proceedings of the Academy of Pedagogical Sciences of the RSFSR], the monthly Sovetskaya Pedagogika [Soviet Pedagogy], and the bimonthly Voprosy Psikhologii [Questions of Psychology]. Since 1963, the Academy has been issuing collection entitled Novye Issledovaniya v Pedagogicheskikh Naukakh [New Research in the Pedagogical Sciences] in order to disseminate information on current research.

A major difference between the Soviet and American conception of educational research is that Russian psychologists often use qualitative rather than quantitative methods of research in instructional psychology in accordance with the prevailing European tradition. American readers may thus find that some of the earlier Russian papers do not comply exactly to U.S. standards of design, analysis, and reporting. By using qualitative methods and by working with small groups, however, the Soviets have been able to penetrate into the child's thoughts and to analyze his mental processes. To this end they have also designed classroom tasks and settings for research and have emphasized long-term, genetic studies of learning.

Russian psychologists have concerned themselves with the dynamics of mental activity and with the aim of arriving at the principles of the learning process itself. They have investigated such areas as: the development of mental operations; the nature and development of thought; the formation of mathematical concepts and the related questions of generalization, abstraction, and concretization; the mental operations of analysis and synthesis; the development of spatial perception; the relation between memory and thought; the development of logical reasoning; the nature of mathematical skills; and the structure and special features of mathematical abilities.

In new approaches to educational research, some Russian psychologists have developed cybernetic and statistical models and techniques, and have made use of algorithms, mathematical logic and information sciences. Much attention has also been given to programmed instruction and to an examination of its psychological problems and its application for greater individualization in learning.

The interrelationship between instruction and child development is a source of sharp disagreement between the Geneva School of psychologists, led by Piaget, and the Soviet psychologists.* The Swiss psychologists ascribe limited significance to the role of instruction in the development of a child. According to them, instruction is subordinate to the specific stages in the development of the child's thinking--stages manifested at certain age levels and relatively independent of the conditions of instruction.

As representatives of the materialistic-evolutionist theory of the mind, Soviet psychologists ascribe a leading role to instruction. They assert that instruction broadens the potential of development, may accelerate it, and may exercise influence not only upon the sequence of the stages of development of the child's thought but even upon the very character of the stages. The Russians study development in the changing conditions of instruction, and by varying these conditions, they demonstrate how the nature of the child's development changes in the process. As a result, they are also investigating tests of giftedness and are using elaborate dynamic, rather than static, indices.

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*See The Problem of Instruction and Development at the 18th International Congress of Psychology by N. A. Menchinskaya and G. G. Saburova, Sovetskaya Pedagogika, 1967, No. 1. (English translation in Soviet Education, July 1967, Vol. 9, No. 9.)

Psychological research has had a considerable effect on the recent Soviet literature on methods of teaching mathematics. Experiments have shown the student's mathematical potential to be greater than had been previously assumed. Consequently, Russian psychologists have advocated the necessity of various changes in the content and methods of mathematical instruction and have participated in designing the new Soviet mathematics curriculum which has been introduced during the 1967-68 academic year.

The aim of this series is to acquaint mathematics educators and teachers with directions, ideas, and accomplishments in the psychology of mathematical instruction in the Soviet Union. The series should assist in opening up avenues of investigation to those who are interested in broadening the foundations of their profession, for it is generally recognized that experiment and research are indispensable for improving content and methods of school mathematics.

We hope that the volumes in this series will be used for study, discussion, and critical analysis in courses or seminars in teacher-training programs or in institutes for in-service teachers at various levels.

At present, materials have been prepared for fifteen volumes. Each book contains one or more articles under a general heading such as The Learning of Mathematical Concepts, The Structure of Mathematical Abilities and Problem Solving in Geometry. The introduction to each volume is intended to provide some background and guidance to its content.

Volumes I to VI were published jointly by the School Mathematics Study Group and the Survey of Recent East European Mathematical Literature, both conducted under grants from the National Science Foundation. When the activities of the School Mathematics Study Group ended in August, 1972, the Department of Mathematics Education at the University of Georgia undertook to assist in the editing of the remaining volumes. We express our appreciation to the Foundation and to the many people and organizations who contributed to the establishment and continuation of the series.

Jeremy Kilpatrick
Izaak Wirszup
Edward G. Begle
James W. Wilson

EDITORIAL NOTES

1. Bracketed numerals in the text refer to the numbered references at the end of each paper. Where there are two figures, e.g. [5:123], the second is a page reference. All references are to Russian editions, although titles have been translated and authors' names transliterated.

2. The transliteration scheme used is that of the Library of Congress, with diacritical marks omitted, except that Ю and Я are rendered as "yu" and "ya" instead of "iu" and "ia."

3. Numbered footnotes are those in the original paper, starred footnotes are used for editors' or translator's comments.

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INTRODUCTION

Joseph R. Hooten, Jr.

The authors of this volume, Menchinskaya and Moro, have accomplished what many have tried: they have created a unique and valuable contribution to the plethora of expositions of "methods of teaching arithmetic in the elementary grades." The reader is cautioned not to allow the all too familiar title to lull him into the false security that this is something he has already seen many times before.

Rather than choose the usual schema for such works (i.e., a detailed analysis of teaching techniques for specific arithmetical instances), the authors have instead chosen what may be best described as a philosophical-psychological analysis of the special problems of teaching mathematics to children. Relying heavily on their own and other Soviet research in early learning, and assuming that the school serves a definite purpose in the Soviet political-social schema, the authors first create a psychological construct, then develop techniques that follow.

A careful reading of the table of contents clearly reveals the fundamental outline of this volume and should quickly indicate to the reader the authors' basic approach. Therefore, to further comment here on the organization of the volume would be redundant. Rather, it may serve a more useful purpose simply to call attention to some of the more interesting sections and to comment briefly on one potential difficulty for the reader.

To take the latter first, the authors frequently refer to a specific problem that occurs in the child's textbook. Unfortunately, the problem is not always reproduced from the student text nor is it always possible for the reader to know the sequence in which the problem occurs. Thus the reader is forced to make his own inference about the nature of the problem. While this makes for difficult reading on occasions, it in no way detracts from the value of the discussion at hand.

The reader is strongly urged to read Chapter I most carefully. While he may find himself in frequent disagreement with one or more of the authors' statements or assumptions, these are absolutely basic to the authors' subsequent positions with regard to teaching methods. Indeed, the authors never once violate their own system.

Perhaps the sections that will most interest American readers will be those dealing with "Mental Calculations" (Chapter III), "Teaching Problem Solving" (Chapter IV), and "Geometry in the Primary Grades" (Chapter V). Chapters III and V give the best insight into the nature of the content of the Russian elementary-school arithmetic curriculum. Some readers may be astonished to discover the degree of mathematical sophistication expected of the children. Others may be a bit surprised to see so much emphasis on getting the correct answer and so little (relatively speaking) on the "nature of mathematics."

Chapter III reveals one of the strongest contrasts between American and Soviet arithmetic curricula. Where the American curricula completely ignore the role of mental calculations, the Soviets consider "mental calculations one of the best means of deepening the theoretical knowledge which children acquire in arithmetic lessons." The authors present a powerful argument for a heavy emphasis on mental calculations based upon their belief that to do so produces a greater knowledge of mathematics. This writer does not recall a similarly powerful argument for <u>not</u> including such instruction in the American curricula. While there are other equally interesting contrasts in this volume, perhaps it is at this point that the most fruitful investigations might begin so as to resolve the issue.

In Chapter IV the authors discuss at some length the research on the source of errors in problem solving, and then build a problem-solving technique designed to prevent such errors. This approach is quite unique and should add considerable breadth to the other available literature on this topic.

The concluding chapter on providing different kinds of instruction for different kind of children not only contains fine pedagogical techniques but also leaves the reader feeling that in the Russian elementary school the uniqueness of the individual is carefully

preserved and gently considered. Not a bad ending for any book that purports to teach adults how to treat a child.

QUESTIONS IN THE METHODS AND PSYCHOLOGY OF TEACHING ARITHMETIC IN THE ELEMENTARY GRADES

by

N. A. Menchinskaya and M. I. Moro

Moscow, "Education" Publishing House, 1965.
Translated by Linda Norwood

CHAPTER I

THE PSYCHOLOGICAL AND DIDACTIC PRINCIPLES OF TEACHING ARITHMETIC

How can arithmetic instruction be made more successful? How can adequate learning be attained in a shorter time?

Knowledge of the principles of instruction (didactic principles), that arise from the goals of instruction and reflect definite laws governing assimilation of study material should be of assistance here. Although educators of various countries and epochs apply the principles familiar to us (conscious learning, use of visual aids, and others), the content varies substantially, depending on the goals and conditions of instruction.

The teacher who is organizing arithmetic instruction should be clearly aware of the principles that will guide him in his work and of the laws governing learning that he should take into account. He also should know distinctly the interrelationship among diverse principles; that is, what place each one occupies in the system of principles.

In disclosing the content of didactic principles in this chapter, we shall give particular attention to their interrelationships, and to the primary significance of some and the subordination of others, which can have various meanings depending on how the main principles are understood.

1. Content of the basic didactic principles and how they are related

In the Soviet school at present, two principles arising from the goals of educating our young people acquire decisive significance: the principle of educative (developmental) instruction and the principle of the relation of education to life outside the school. The goal formulated in the program of the Communist Party of the Soviet Union--to ensure the development of the well-rounded personality--can be attained only if instruction is carried out with maximum

educational effect. That is, the instruction must facilitate not only the acquisition of knowledge, not only mental development, but also development of the personality as a whole, the formation of Communist morality, and dialectical-materialistic world-views and convictions.

In June, 1963, the plenum of the Central Committee of the Communist Party set the task of overcoming the gap between training and instruction that still exists in our schools. This signifies a need to realize first of all the principle of educative instruction.

We should bear in mind that the principle of "educative instruction" has held a large place in the bourgeois science of education. It was linked with the name of the German educator, Johann Friedrich Herbart. However, this principle was understood quite differently: not the all-round development of the personality of a free man but, on the contrary, discipline through crushing the will, compulsion--this was the aim of education according to Herbart.

It would be wrong, however, to talk about educative instruction in Soviet schools without regard to the principle of its relation to life and the need for unifying theory and practice.

The purpose of education is to instill in children both a desire to study so that they can do their share in laboring for the common good, and socially useful skills and habits. Naturally, the question may arise, what are the possibilities for realizing these goals at the elementary level? These possibilities are considerable. To realize them, it is necessary to take into consideration certain other principles which more concretely determine how to teach. We have in mind, above all, the well-known principles of learning, and practice of instruction. If the teacher is able to effect them consistent with the principal requirements formulated above, then he can be assured of success.

First let us show why these two principles are closely linked by showing that one necessarily presupposes the other and that this relationship is founded on a definite law of learning. The student can

attain full awareness of mastery only when he does not passively take in new material, but uses it actively. A generalization of some law of arithmetic such as the commutative law of addition will be learned consciously when the student himself has discovered, from solving suitable addition examples, that the result does not change when the addends are transposed.

Activity has a positive influence not only upon the thinking processes (when the student is required to master concepts and laws), but also upon the memory processes, increasing the capacity to remember and to reinforce what is remembered.

Try a simple psychological experiment on your pupils and you will be convinced of the existence of this principle. Without warning, tell them to produce (name) the numerical data in the condition of an arithmetic problem they have solved. Perform two versions of this experiment: in one, use prepared numerical data, as is usually done in problem solving; in the other, tell the children to make up numbers themselves. The experiment, as a rule, will show that they remember the numbers they thought up themselves better than the ones which they were given.[1]

Thus, an opportunity to work actively with the material is the most important condition of conscious learning. This does not, however, exhaust the requirements for conscious learning. To learn consciously means to understand what learning is for and what the possibilities are for applying acquired knowledge in practical activity.

Again we see the important role of the student's active application of his knowledge in his own practice, academic and occupational. But the student's comprehension of why arithmetic is studied cannot be determined merely by his own practical activity. Extensive social practice, that is, using mathematics in life, gives children a motive for studying and brings about a desire to study mathematics. Thus we see that the principle of the connection between instruction and life

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[1]This experiment was performed by the psychologist P. I. Zinchenko [76].

directly determines (in the conditions of our schools) the content of the principle of conscious learning. Not only the content of the knowledge acquired but also the very methods of its acquisition and use must be motivated. In the teaching process we must arouse in the children a longing to perfect their methods of calculating and solving problems and to replace less precise ones with more precise and economical ones.

In practice we often find that the primary schoolchild tries to avoid difficulties, preferring to use more conventional methods. For example, some children, after mastering the techniques of written calculation, begin to use these methods even in solving mentally. And if they are not allowed to write, they "write down" figures mentally ("To add 9 and 2, I write down 1 and carry 1"). What makes these children resort to such a cumbersome method? The desire to act according to conventional rules, thereby avoiding active mental effort. They have not adopted the practice of deviating from the beaten path and searching out more rational methods.

A most important task of mathematics instruction is to give schoolchildren the impetus to think actively, to surmount difficulties in solving the various problems encountered in life, and to search for the most rational ways of solving these problems. The arousal of such a desire is inseparably linked to the formation of stable "cognitive" interests (interests linked to the process of cognition of the science being studied). But here again we encounter the principles of active practice and conscious learning in their intimate relationship. In this encounter we discover a new aspect to these principles.

Conscious learning presupposes not only the work of the pupil's mind but also the direct, active participation of his will and emotions. It assumes formation of a definite emotional attitude toward the learning process. "Without 'human emotions,' there is not and there cannot be a human search for truth," wrote V. I. Lenin [34]. To some extent this tenet is applicable to our pupil, for whom learning must become, to a certain extent, the process of "procuring" knowledge.

The greater the teacher's skill, the better he can organize the pupils' process of procuring knowledge; not only is he communicating the requisite sum of knowledge, but he is also forming an active member of the future Communist society. Under these conditions the pupil not only gains knowledge in the process of instruction, but becomes proficient in methods and techniques of procuring and applying knowledge.

It is these two elements which are necessary in order for instruction to be developmental. As we know, 'an empty head does not reason" (P. P. Blonskii), but a person may have much knowledge that remains a "dead weight" if he is not able to expand and use it. This kind of ability should be systematically formed in young children. This is considerably more difficult for the teacher than merely imparting the knowledge stipulated by the curriculum.

In this book we shall give particular attention to the problem of forming the skills of acquiring and applying knowledge in arithmetic. We shall try to show how these skills are formed and manifested in various kinds of school activities--when mastery of new material is required in connection with solving various examples and problems.

It is especially important to discover which teaching conditions (organization of the lesson, different forms of independent work, and the like) facilitate the formation of useful skills, and which retard their development. We should keep in mind that skills differ depending upon whether they are limited and specialized or whether they are more general, appearing in various types of activities. For example, the ability to add numbers correctly in performing written calculations is a limited, specialized skill. The ability to analyze (break down), synthesize (unite), and generalize--that is, to carry out a particular mental operation--belongs to the other category. If such a skill has been formed, it may show up in different kinds of work, both in the deduction of a new rule of arithmetic and in the solution of examples and problems. Finally, there is a category of more general, broader skills--knowing how to organize one's procedure, planning the work, checking it systematically, finishing what has been started, striving for accuracy

and thoroughness of execution, and so on. Such skills may be manifested not only in arithmetic class, but in other kinds of schoolwork and vocational activities as well.

The teacher's efforts should be directed at developing a complete system of such skills in the children--both the narrower, more particular skills, and the broader, more general ones. Now let us briefly describe what "activating a skill" means and what is needed for it to occur.

For the primary schoolchild to be active in the learning process, first he needs plenty of opportunity to display independence in schoolwork; second, a knowledge of effective techniques of working independently; third, the arousal of a desire for independence through the creation of a suitable motivation. In other words, an independent, creative approach to problem solving must become vital to him.

Recently the principle of the activation of learning which calls for awareness and practice of what one learns, began to be utilized rather extensively, and to have a decisive influence on the understanding of such well-known principles as accessibility, the systematic approach, and visual presentation in teaching.

In terms of the activation of learning, accessible material means material that the student can master through unassisted mental effort (under the teacher's direction, of course). This is an essential point in understanding accessibility. One should not only avoid excessively difficult material but also material that is too easy for the student to master, in which everything is clear to him from the very beginning and in which no problems or questions demanding mental effort arise.

Systematic instruction also takes on a somewhat different meaning when seen in terms of the activation of learning. The principle that a definite sequence is necessary in assimilating knowledge and forming skills undoubtedly remains in force; to master more complex knowledge and skills, one must master the simpler ones on which they are based.

At the same time, a different method may be fully justified in the activation of learning. For example, in teaching children in the first grade how to solve problems, the question arises how best to pass from problems in one operation to compound problems in two operations.

Two different ways can be advanced. In the first, one should proceed from simple problems and, combining two problems into one, show the peculiar character of the new compound problem. In the second, one should tell the children to solve a two-operation problem that is new to them, and to break it up into two simple problems while solving it.

Research shows that the second way is more effective. It is easier, when using this method, for children to realize the specific character of a compound problem compared to a simple one. Why? How can this be explained? In the second case the teacher confronts them with a difficulty immediately, giving them a new kind of problem and explaining that it cannot be solved immediately in one operation, that is, in the usual way. In the first case, however, this confrontation with a new difficulty is lacking. The children gradually pass from simple problems they know to a new problem created or composed on the basis of the old ones.

When deciding a question of instructional methods involving ways of introducing certain material, one should be flexible and not always follow the same path (from the easy to the more difficult). It is often expedient to present children with a new difficulty so as to make them aware of it, and so that later it will be easier for them to use this new material. In the example cited, the difficulty children encountered in solving their first two-operation problems will help them to handle problems of unfamiliar form later.

Activation of learning fundamentally influences understanding of the principle of visual presentation as well. "From things one must go to ideas," Jan Amos Komenský, Czech educator, said. Since then this principle has been extensively elaborated in education. In particular, in the elaboration of arithmetic methods it has received much attention.

As instruction proceeds, the child gradually masters certain abstract concepts, but apart from those which have been assimilated. New, even more abstract ones, are introduced, and again it becomes necessary to make their content concrete by using various means of visual presentation. Even today this understanding of the principle of

visual presentation remains unchanged, and Soviet teachers are applying it fruitfully. But one aspect of it which should be particularly stressed when the task is to activate learning, has not yet found sufficient reflection in practical arithmetic instruction. We mean the independent use of visual aids by primary schoolchildren. Until now little attention has been given to this essential practice. Students have been given definite visual aids to use in calculation; and after a certain amount of practice with them, the teacher takes them away. If they had a difficult arithmetic problem, however, the teacher would make a diagram that would help them understand the relationship between the data and unknown contained in the problem's condition. But the question arises: Under these conditions, will a child learn to use visual aids in independent work? This question has to be answered in the negative. Under these conditions this skill cannot be formed because the child lacks practice in using visual aids by himself. The children themselves should imagine clearly what the problem is talking about without any special prodding from the teacher. Students in the upper grades of primary school (third and fourth grades) are able to construct a diagram that helps them to find the relationship between the problem's data and the unknown.

In a number of cases it is enough to make a problem given in abstract form more concrete, without resorting to pictorial visual aids. For example, in a special experiment, a third-grade pupil was asked how the remainder varies if the subtrahend is increased. The boy could not give an immediate answer. He had not yet studied the law defining the interrelationships of these components. However, he handled the assignment successfully after independently giving it concrete substance by translating an abstract problem into one of practical value. "If I had 5 rubles and they took away 2, then I'd have 3 rubles; and if they took away more of my money--4 rubles--then there'd be less, only 1 ruble." The boy then knew how to formulate an answer in abstract concepts: "If the subtrahend is increased, the remainder decreases."

Thus we have shown that didactic principles are not simply the sum of prerequisites for success in the learning process. They comprise a system whose elements, although linked inseparably, are not equivalent. They play unequal roles in this system. Some of them, such as the principle of educational instruction and its connection with life, have decisive significance. Through the principle of activation it affects all the other principles. Others, such as the principles of accessibility, the systematic approach, and visual presentation, have secondary significance, and their content depends in substantial measure on whether the principle of activation of learning is realized.

In characterizing the principle of activation of learning, we have shown that while it arises from the teaching aims of the Soviet school, it rests at the same time on a definite psychological law governing the mastery of school material.

However, still other laws govern learning, which the teacher must know in order to have better control over them. We shall describe them below, giving attention to what is required for adequate learning and application of knowledge (and what demands are made on a pupil's thinking in the learning process) and to the conditions under which it is easiest to attain these requirements.

2. The link between analysis and synthesis in learning arithmetic

In mastering laws and concepts of arithmetic and in solving problems and examples, schoolchildren are continually carrying out the basic mental operations of analysis and synthesis. At first they have poor control over these processes, and only gradually do they form the ability to analyze and synthesize.

Each of you can observe in yourselves, manifestations of the laws of analysis and synthesis. Imagine that some unfamiliar apparatus or mechanism is being demonstrated before you. You perceive it in its most general, unbroken form. Neither its purpose nor its working principle is clear to you. At this first stage in the examination of an unfamiliar object, you perform an elementary yet still imperfect synthesis which is not based on analysis. Then in examining the object, you

separate its individual parts and interpret their purpose, drawing upon what knowledge you have--i.e., you perform an analysis of the object. On the basis of this analysis an integrated idea of the object is synthesized, which makes possible a definite surmise or speculation as to its function. The synthesis which you are now performing is more complete, since it is based on analysis.

The success of any intellectual work depends on the completeness of analysis and synthesis, on a person's ability to effect these two processes in close association with each other.

In the process of learning arithmetic, a certain gap is unavoidable in the levels at which children carry out both operations. This gap is temporary, and whether it is overcome in as little time as possible, whether the children are taught to perform closely-linked analysis, depends wholly on the teacher's skill.

We can find a most striking example of the temporary disunity of analysis and synthesis in an experiment with pre-school and nursery school children in forming their first numerical concepts. The following was commonly observed:

> A mother held two sticks in her hands and asked her two-year-old girl, "How many sticks do I have?" "1, 2, 3, 4, 5" was the answer. When the question was repeated, the child again recited numbers, without giving any designation to the whole aggregate: "1, 2, 3." But immediately afterward, although she had not been asked "How many," the little girl said to her mother, "Give me the two sticks." Seeing two butterflies flying together, she cried, "Two butterflies!" But a few minutes later, when the question followed, "How many are are there?" the answer was different: "1, 2, 5, 7."[2]

What do these observations say? At a definite stage in a child's development, in connection with forming his numerical ideas, two imperfect forms of pre-numerical concepts co-exist. One form reflects primary synthesis, not based on a preceding analysis. "Two," says the child, relating this to a definite quantity, without having previously

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[2]Case taken from the observations of Menchinskaya [38].

broken it down into individual units, without counting. The second form, on the other hand, amounts to elementary analysis, since the child enumerates a consecutive series of numbers ("1, 2, 5, 7"). However, this analysis does not lead to a subsequent synthesis, since this enumeration is not followed by naming the last number as the designation of the total result of counting. Furthermore, it must be kept in mind that the analysis in itself is extremely imperfect in the initial stages, since a unit in a numerical series still does not correspond to an element in the aggregate of objects being counted, and the child makes an error in reproducing the series of numbers.

Gradually children become proficient in this complex operation of analysis, but the concept of number is formed only when analysis and synthesis are implemented in close relation to each other. Breaking up the sum total into units and accompanying this with a recitation of a series of numbers leads to subsequent unification of the given quantity's elements, naming the last number in the series as the total quantitative result.

At all stages of instruction, attention should be directed at forming the ability to carry out analysis and synthesis, implementing the link between them. A child cannot form a single concept without executing analysis and synthesis. Comprehensive analysis has great significance here. Incomplete analysis inevitably makes for erroneous synthesis. This phenomenon is especially striking in solving arithmetic problems.

First and second graders frequently make mistakes like the following. In choosing an arithmetical operation, they do not consider the question, but perform whatever operation the numbers themselves "prompt" them to. For example, to solve the problem: "Vanya had 2 apples and Petya had 3. How many more apples did Petya have?", some of them mistakenly add 2 and 3, making this mistake simply because they do not analyze the text of the problem and do not single out the question, even though it was needed for choosing the right operation.

Often in choosing an arithmetical operation, primary school-children tend to rely on a specific word taken out of context, in isolation from the rest of the problem. For example, in coming across such words as "flew away" in a problem's condition, some pupils subtract, even though the presence of these words in the text does not always indicate subtraction. (For example, the problem: "Some birds were sitting on a branch. Three flew away and 2 were left. How many birds had there been on the branch?" contains the words "flew away," but it is solved by adding, not subtracting.

Again, this kind of mistake results from incomplete analysis. However, in speaking of complete analysis of facts and phenomena, one should stress that this analysis must be subordinate to a definite problem. Any complex phenomenon has many different aspects. Depending on the problem, one should sometimes single out one aspect, sometimes another. When we are dealing with a concept, it reflects not all, but only the most significant, aspects of objects and phenomena. Various aspects may function as essential, depending upon the concepts. For example, in forming the concept of an integer, a child deals with quantitative correlations; from the standpoint of this concept, the size and form of the objects being counted are not significant. But in geometry, form is first and foremost, and other characteristics are no longer relevant. Thus, the task of teaching is to train children to perform purposeful analysis which takes into consideration the goal of the assignment.

Mathematics requires children to discern relevant characteristics inherent in a whole series of facts and phenomena and to generalize them, formulating suitable concepts and laws. How can the formation of correct generalizations be ensured? We shall answer this question in the next section.

3. Principles determining the formation of correct generalizations

The question we have raised is very important. It is difficult at first for children to discern and be aware of the characteristics which are relevant, thereby separating them from numerous irrelevant

aspects of facts and phenomena. Frequently these insignificant characteristics are expressed visually and the child sees them first.

Try giving a child who has not yet formed numerical notions the same number of objects (for example, five buttons), but arrange them differently. First arrange them in a small circle, then make a figure of extended length. If you ask the child which group has more buttons, he will point to the one which occupies more space. Thus the arrangement of the objects in space determines his judgment; and the child does not notice quantitative relationships beyond these visually expressed characteristics.

To make discernment of relevant features faster and easier, the children must repeatedly convince themselves that the same quantity can have different spatial arrangements. Children find it especially convincing to change the spatial arrangement of a definite number of objects themselves. For example, they can be told to put five buttons that are arranged in a circle onto a ruler, thus extending them in length. Of similar significance are first-grade assignments in counting and establishing the equality of quantities made up of different objects or objects of the same kind but varying in size, color, and the like.

It is very important for the mathematical development of primary schoolchildren that they understand that the results of the numerical operations they perform (addition, subtraction, and others) do not depend upon the names of the objects. If children add 5 matches and 3 matches, for example, they will get the same numerical result as by adding any other objects (sticks, blocks, etc.).

Whether the children have formed this necessary generalization can be checked very easily. First, tell a student to add 3 matches and 5 matches. When he gives 8 as the result, ask him what happens if 3 pencils and 5 pencils are added. The child who has formed the generalization will answer, "That will be 8, too," or "Still 8." The one who has not yet made the generalization will want to get pencils to perform the numerical operation anew. (In any such case the choice of numbers depends on the limits beyond which the child can no longer compute without using objects.)

To form this generalization it is necessary to demonstrate visually to the children that the result does not change although the names may be the most diverse. They should repeatedly convince themselves of this with their own experiments.

Exactly the same requirements for forming generalizations hold true in other sections of the arithmetic course (the study of geometric material, problem solving), but only in these cases do the children have occasion to handle other relevant and irrelevant features.

Errors in generalizations similar to those described above may be encountered not only in first and second grade, but in third and fourth grade as well. For example, some pupils cannot recognize a rectangle extended in length since they have the mistaken idea that the ratio of the lengths of adjacent sides is a criterion of the given concept. This erroneous conclusion would not have been made if the children had seen and constructed, for themselves, rectangles with different side ratios.

In the solution of arithmetic problems, mistakes in generalizations occur in various forms. Either, as noted earlier, children consider one definite word (most often a verb) to be invariably linked with a definite operation when they are choosing an arithmetical operation, or they regard some verbal expression as most significant in determining the type of problem. For example, after getting to know the type of problem in which two numbers are found from their sum and ratio, some children think it is enough to notice the words "so many times as much" in the problem's condition in order to decide that the problem must belong to this type.

Again in these cases, a necessary condition for correct generalizations is demonstrating with concrete examples that features which can change drastically are not relevant for a certain concept. For example, one may encounter rectangles with different proportions of lengths of adjacent sides, or a ratio may occur in different problems, typical and atypical; it may have a different meaning depending on what other data it is combined with in the condition. Thus we can now formulate a general rule: A necessary condition for instilling

correct generalizations in children is the variation of inessential features of the material presented, keeping the essentials constant and unchanged. Here it is important that the children actively vary the inessential parts of the material themselves when illustrating a concept or law.

The teacher should have sufficient material available for exercises. The children can be asked to substitute names of the objects subject to calculation while performing the same numerical operation, draw rectangles of various sizes and with various side ratios, think up problems in an arithmetical operation using different verbs in the condition of the problem, and so on.

It is of great importance for the students to identify not only the relevant characteristics (or the principle), but also the irrelevant (variable) ones. The former are usually given in the textbook definitions, where they are specially emphasized. Irrelevant characteristics usually are not specified. Still, it is very important that children know how to express vocally what characteristics are irrelevant for a given concept (or principle).[3] Here, of course, we do not have in mind any complicated statements. It is quite enough if first graders say, "It's the same no matter what we count." Or similarly, while drawing rectangles, they might say, "Rectangles are different--big and little--and their sides can be different." But here, one important requirement must always be observed in teaching procedure: The definitions children give of concepts, their statements of laws, and enumeration of irrelevant features, should always arise from their own experience, and be a result, not a condition, of their work with the material so that they are provided with the opportunity to become fully aware of them.

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[3]This requirement was formulated on the basis of a series of investigations by Kabanova-Meller [25].

One should always remember Ushinskii's dictum which vividly expressed this requirement: "A word is good when it faithfully expresses an idea; and it expresses an idea faithfully when it grows out of it, like skin from an organism, and is not put on like a glove sewn from other skin [68:34]."

Even now a dispute is going on among methodologists and teachers over whether it is necessary to give the children the names of problem types. We shall easily resolve this dispute if we proceed from the requirement set forth above. Naming the type is useful when it results from the child's comprehension of the characteristic feature of the type-problem, when he himself makes a general conclusion. Conversely, the name may only do harm if the teacher communicates it prematurely and reduces it to some tag, related to the purely external features of a type-problem.

Let us consider in greater detail what the process of generalizing represents in mastering arithmetical concepts, as well as others. The child learns a general proposition because he compares observable facts and phenomena and reveals the similarities and differences in them. We see from the previous examples that discernment of the general (ratios, object form, problem type, etc.) is based on ascertaining similar features inherent in a number of phenomena. At the same time, distinctions in these phenomena are noted, allowing the relevant features to be separated from the irrelevant. In counting identical quantities, when children say, "Five white balls and five red balls," they are noting a similarity in one essential respect, while keeping in mind that this resemblance is inherent in different phenomena.

Comparison is the mental activity which the child constantly implements in learning. Furthermore, Ushinskii [68] pointed out that comparison is the basis of all thinking. Success in learning is determined, to a significant degree, by whether the pupil has formed the ability to compare, i.e., to notice the similar and the dissimilar.

In mathematics instruction, even in the primary grades, pupils are required to perform complex forms of comparison. They have to note what is similar in phenomena which outwardly differ greatly, and at the same time, they must discern a difference where the resemblance is strong.

A significant number of mistakes that pupils make in arithmetic occur because they do not know how to compare; instead, they operate by analogy (in solving examples and problems) when a change in operation method is required and, conversely, fail to use known methods where they should, since they do not notice the similarity.

The little words "na" and "v" in the expressions "na skol'ko" ("How much" larger or smaller) and "v skol'ko raz" ("how many times" larger or smaller), which require fundamental changes in operational method, are often overlooked by the pupils, and mistakes inevitably occur.

One stumbling block for children is the difference between two systems of names in solving problems of division into parts and division by content. In both cases one has to write the concrete name that accompanies the dividend; but in the first, it is the divisor that is written without a name, and in the second, it is the quotient. Quite frequently children do not notice this difference and, in writing down the answer to a problem, liken one system of notation to the other.

Even in type-problem solutions one often encounters similar mistakes. For example, the pupils correctly solved problems in finding two numbers from their sum and ratio, but when they were given another problem whose terms indicated the difference and the ratio, they failed to notice the important distinction in the conditions and did the problem incorrectly by adding the parts.

At the same time, the reverse is often found in the classroom. Pupils successfully solve a problem in division into parts, but through a definite verbal formula (when the conditions say "how many times more"). If the wording is changed (for example, "If we divide one number by another, how much will the result be?"), a number of students perceive it as a significant change and fail to use a method they know well.

What are some ways of preventing this kind of mistake, and how can one combat it once it has appeared? In particular, how is one to counteract mistakes of confusing similar facts and circumstances?

One general law of mental activity is well known. Concrete, contrasting phenomena are most easily distinguished. This law applies with equal validity both to perception and to thought processes. It is based on the physiological law widely taught by Pavlov and his pupils: of a number of stimuli acting on the nervous system, contrasting stimuli are the first to be differentiated. In the study of concept formation at the preschool and early school age, we constantly meet the same fact. Contrasting concepts, one aiding awareness of the other, are formed first of all: "big--little," "good--bad," "many--few," "long--short," and so on. Such concept pairs are easily learned because they are in contrast to each other.

A mathematics course, particularly a primary-grade arithmetic course, contains many pairs of contrasting concepts: the operations of addition and subtraction, multiplication and division, or a direct problem and its reverse. In many cases (if not in the overwhelming majority), however, reversible concepts, operations, and problems are studied in school at different times, and the study of them is often separated by rather long intervals. Whether this hampers learning is a question we shall answer a little later.

Psychological research using various materials (curricular and extracurricular) shows that contraposition of distinct concepts and rules prevents subsequent confusion of them. There are two ways of contrasting--simultaneous and consecutive. In the first, both concepts (or rules) are introduced at the same time (in the same lesson) in contrast to each other. In the second, first one of the concepts being compared is studied, and when it has been mastered, the second is introduced, on the basis of its contrast to the first.

In teaching arithmetic, what pairs of contrasting concepts can expediently be introduced simultaneously, and what concepts should be contrasted consecutively? Can it sometimes be expedient to make

the contrast only after both concepts have been mastered? Up to now this third way has been most widely used in school. Educational research is still unable to answer these questions definitively. Further experimentation on the effectiveness of the three different ways of teaching, in conformity with various sections of the curriculum, needs to be organized.

Even now, however, one can assert the following in all certainty: Contrast creates undoubted advantages for learning, since both concepts or rules are included in a single system of knowledge, and learning the features of one concept reinforces the mastery of the features of the other. Thus one may infer, for example, that it would be expedient to introduce addition and subtraction simultaneously.[4]

But when the concepts or principles being studied represent a rather complex system of features, and not all features of this system are diametrically opposite each other, it is more expedient to contrast them later. This method was used, for example, with respect to the two kinds of division--division into parts and division by content.[5] The difficulty in solving these problems consists in the different ways of writing concrete names, as noted above. The notation of the dividend is the same in both cases; the difference in the methods are in the two other components. In compliance with the curriculum, the division into parts problem was studied first, and the discovery by content was introduced on the basis of contrast with the first. Typically, the pupils themselves discovered what peculiarities distinguished the new problems from problems on division into parts.

Therefore, when a new assignment recalls an old one in certain respects, inducing students to apply already familiar operational

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[4]Such an attempt is being made, in particular, by Erdniev and a number of Stavropol teachers [17, 18].

[5]N. I. Kozheurova, a teacher, ran the experiment with the assistance of Menchinskaya [37].

methods, it is essential to stress the difference between the new assignment and the old one and to establish its specific character from the very beginning of the assignment. The third way, contrasting both concepts or rules only when an interval has elapsed after they have been studied separately, is absolutely inadmissible in cases of this kind. Thus contrasting at different stages should help prevent confusion of similar concepts and rules. Contrasting needs to become a systematically applied method in academic practice .

Sometimes it is advisable to supplement the present curriculum in those sections which limit opportunities for contrasting. This observation is made in reference to type-problems. The curriculum introduces one variety (for example, problems "on the sum and ratio") and excludes another (for example, problems "on the difference and ratio"). Lack of opportunity to change the type-method as the problem's condition changes not only robs type-problems of cognitive value, but also does harm by preventing the pupils from thinking, since it is enough for them to "adjust" the problem's solution in a particular pattern to make it successful.

4. Abstraction and its role in the process of learning and applying knowledge. The link between concrete and abstract thinking in learning arithmetic.

In the sections devoted to the problem of generalizations and conditions for forming them, we have touched upon the problem of the asbtraction process, since mastery of concepts and laws is impossible without distinguishing certain characteristics and abstracting them from others. To discover what claims practical schoolwork demands of the abstraction process, what kinds of abstraction exist, and what conditions facilitate successful completion of the process, it is necessary to reconsider this question specifically.

In learning arithmetical concepts and laws, students unify similar, relevant features inherent in a number of phenomena, separating them from the other, irrelevant ones. Generalization and abstraction are performed together, inseparably linked.

But in academic practice there is more than this one form of

abstraction. Another significant form of abstraction uses previous knowledge to solve a problem . That is, the child must find a familiar concept (or principle) in an unfamiliar, concrete problem. The difference between these forms of abstraction is quite easy to grasp if they are compared by using an example taken from arithmetic instruction. To form the concept of a problem type (for example, involving "the sum and ratio"), you show how the given type problem is special by giving several examples (with the active help of the children). The characteristic feature of the condition and the appropriate methods of solution, which are identical for the series of homogeneous are problems emphasized, (for example, the problem's condition gives the sum of two numbers and their ratio, and the typical method is to introduce a contingent "part" and then to add the parts).

The children notice what the problems in this category have in common--an essential step in solving them--and omit irrelevant features (different subject, different numerical data, etc.). In other words, when classifying the problem according to a given type they do not consider insignificant features. Here we are dealing with the first kind of abstraction, which is implemented together with generalization.

At later stages of instruction you tell the students to solve a problem of this type on their own. You naturally make various changes in the irrelevant aspects of these problems--you change wording, subject, numerical data, or you introduce an additional condition.

This psychological task that confronts the child is to recognize in a given arithmetic problem a type of problem studied earlier. To do this he must distinguish its essential features, "purifying" them of the irrelevant features that "cloud" or "mask," so to speak, the features of the problem type that was studied. Here we are dealing with the second kind of abstraction, which acquires the significance of an independent process since it is not implemented together with generalization. The child has already formed the generalization. He only needs to find a general principle, articulating it from new concrete conditions.

The difference between these two kinds of abstraction has important consequences which dictate a different teaching approach. A pupil's difficulties in carrying out the first form of abstraction (let us call

it the "primary" form) are lightened when the second form of abstraction (let us, accordingly, call it the "secondary" one) is implemented. Indeed, when we are forming a concept of a type of problem, we need to begin with a more concrete problem and then have the children go on to an abstract problem. To fall back on an example used earlier, we should first show "the sum and ratio" type of problem with real objects, then move on to a topical descriptive problem:

> Two boys had 12 apples. One had 3 times as many as the other. How many apples did each boy have?

and only then give them a problem with abstract content:

> The sum of two numbers is 24, and one number is twice as large as the other. What are the numbers?

By contrast, when we are dealing with secondary abstraction, introducing specific, topical details into a problem can only make it harder for the pupil to solve, since he has to recognize a familiar mathematical structure in a more complicated concrete phenomenon, discarding many inessential details. In one investigation this fact was demonstrated in an example of solving problems by "equating data." The fourth graders successfully applied the equating method to solving problems in which they were directly told the different prices of two groups of objects. For example, the problem:

> Three books and 5 notebooks cost 40 kopeks, 1 book and 1 notebook cost 12 kopeks. How much each do a book and a notebook cost?

But when they were given a problem to which more specific conditions had been added:

> A girl bought 3 books and 5 notebooks and paid 40 kopeks the first time. The next time, she bought 1 book and 1 notebook at the same prices and paid 12 kopeks for them. How much each do a book and notebook cost?

many of the same pupils could not solve it. The cause of the difficulty was that in this problem, the interrelation of the prices of the two groups of objects was "clouded" by new, specific, purely topical details: "the first time," "the second time," "bought," and "paid." The necessity of abstracting from a large number of mathematically insignificant details complicated the solution of the problem.

Consequently, a problem varies in psychological difficulty (for the pupil) depending on whether we are forming a concept (principle) or giving exercises in recognizing a familiar concept (principle) under new, concrete conditions. This results in a different order of presenting problems from one case to the other.

In approaching the two forms of abstraction in solving arithmetic problems, there is another highly significant consideration. When primary abstraction is carried out, one should ensure fairly extensive experience by acquainting the pupils with various instances of the concept being studied. In secondary abstraction, the diagram, which aids in the selection of a general principle under new, concrete conditions, should play a decisive role.

Psychological research has revealed the positive role of diagrams in problem solving [6]. It is well known that the basic difficulty pupils have in solving arithmetic problems is the gap which exists for them between the concrete story aspect of the problem and the abstract mathematical relationship expressed in it. A diagram (expressing in abstract spatial form a relationship between data and unknown—for example, in the form of rectangles of various lengths) helps to overcome this gap by uniting visual elements and abstract material. A diagram, on the one hand, helps to eliminate the problem's specific subject since it is more abstract (outside the subject). On the other hand, it facilitates awareness of the abstract mathematical relation, since the diagram itself expresses this relation, in more visual, graphic form.

From this point of view it is quite typical for the diagram to be unable immediately to help overcome the gap between the specific and the abstract in the pupils' minds. An important condition for using it successfully is to compare it with a more concrete sort of drawing--a picture illustrating the interrelation of the objects described in the problem. In this instance a gradual abstraction from the irrelevant facets of the problem's condition takes place. In analyzing a picture, the pupil is distracted from details of the story and directs his attention to relating the objects. In analyzing a diagram, he is even distracted from the objects themselves and concentrates only on the relationships of the quantities expressed in the condition.

In the course of this exposition we have been approaching formulation of a most important principle of instruction in general, and of mathematics instruction in particular: the principle of the interaction of concrete and abstract thought in the learning process. The significance of this principle (and the law it reflects) is clearly revealed in the material we have considered.

We see that a necessary prerequisite for forming abstract concepts is the accumulation of concrete experience, including acquaintance with different phenomena. Recognition of a general, abstract principle will come about only with the conquest of the gap between the concrete and the abstract which initially exists in the mind of the person who is solving the problem.

At this point we should remember that the concepts of "concrete" and "abstract" are relative. The same academic material may be abstract for a pupil at one stage of instruction, and at the next stage become concrete, since its role in the learning process has changed and it begins to function as a concrete support for new, more abstract material.

For example, at the initial stage of instruction, in which pupils advance from practical operations with quantities of objects to arithmetical operations, the numbers they work with and the calculations they perform are abstract. But to the pupils, these numbers and the operations with them become a distinctive, concrete support for studying the arithmetical operations and for solving problems.

You can easily discover this for yourself if you give first graders who are just beginning to solve arithmetic problems a problem without number data. Many of them will not be able to solve it, since they lack the basic support of operations with numbers. Operations with numbers play an analogous role (of concrete support) in the transition to more abstract material in the upper grades--e.g., manipulating letter symbols in algebra.

The role of concrete material varies not only with the stage of instruction but also with the problem. In mastering new concepts and principles, concrete material is a support to learning. But in recognizing a previously studied concept or principle under new conditions, concrete material (as was shown in detail in the example involving solution of type-problems) becomes a condition which not only fails to simplify but actually

complicates an assignment. In both cases concrete material is useful in school, but in the second case it will play a role very different from that in the first case, as we usually imply when discussing the principle of visual demonstration.

The prior use of visual aids should be considered in the light of the broader problem of coordinating the concrete and the abstract in the teaching process.

We must turn our attention to still another facet of visual aids. They are certainly not always useful in teaching. Often in the classroom we find them misused, when they are applied unnecessarily. Sometimes they can even play a negative role, diverting us from the problem at hand. Many cases can be cited. For example, a first grader is being taught how to choose an arithmetical operation (addition or subtraction) in solving arithmetic problems, and to this end the teacher brings out a picture of birds, some sitting on a branch and some flying toward them (or away from them). By looking at this picture the pupil sees the birds that are to be counted. Obviously he need not mull over what operation to use when he can tell immediately how many birds there are after the others have joined them (or have flown away). Here the visual approach does not facilitate formation of the ability to choose an operation to solve a problem, but hinders it.

A picture plays such a role when it is used at the initial stages of teaching children to formulate problems. Keep in mind that at first children find a basic difficulty in putting a question into a problem. It is often observed that when first graders are told to think up a problem, they make up one without a question, or instead of a question they immediately give the answer. So it is especially important to have the pupil construct problems by himself in order to teach him how to formulate a question. But what is gained when a picture is used as the basis for formulating a problem? What happens is just the opposite of what the teacher wants. For example, children see in a picture that a little girl picked 3 mushrooms and a little boy picked 5. Naturally they see no reason to put into words the question, "How many mushrooms in all do the boy and girl have?" in their problem. Instead of a question they give a numerical answer, "They picked 8 mushrooms in all." In this case there is no need for a question, since the result is immediately obvious.

Such unjustified use of visual aids occurs at various levels of

instruction. For example, in arithmetic books for the various grades, one can find illustrations of the things described in the problems even though the children are well acquainted with these things, and looking at them can only lead their thoughts astray. So how can they concentrate on the problem's mathematical structure?

In practice, one visual form, the illustrative form (pictures), is often misused, while analytic types of pictorial visual aids--diagrams and sketches--are strikingly underrated. A significantly larger place in arithmetic teaching should be given to these forms. The goals of polytechnical instruction demand it. From this viewpoint, development of children's spatial ideas, formation of the ability to read and construct a drawing (draft) both mentally and on paper, the ability to transform drawings and their separate elements mentally, and the like, acquire great significance [5: Chapt. VI]. Only if we make sure that the children have enough exercises of this type will we succeed in accomplishing the necessary development of their concrete thought in the primary grades, imperative for their subsequent success, not only in academic fields but in other work as well.

It should be kept in mind that concrete thinking, the basis for developing abstract thinking, develops by itself in the course of schooling, at all ages.

We stress these propositions because until now a simplified idea of the development of children's thought has been popularized and reflected in arithmetic teaching methods, especially those that have to do with the visual aids principle. The idea comes down to this: A small child thinks in concrete terms; therefore in teaching him mathematics, one needs to use visual aids extensively. Furthermore, to the extent that he will master abstract concepts and rules, it is necessary gradually to remove visual aids, going from full to partial use of them and then removing them entirely.

What is correct and incorrect in these assertions? It is correct that in the early stages of growth a child is capable of learning abstract material only through the concrete or the visual. It is also true that visual aids, so necessary in the early stages of acquiring abstract knowledge, are unneccessary later, after playing the role of temporary support.

But in what was said above an essential idea has been left out. Both abstract and concrete thinking develop with age. A small child's concrete thinking is developed to a very limited degree. So visual aids are not only necessary to facilitate development of abstract thinking, but also (to no less a degree) to form various aspects of concrete thinking. In connection with this, certain visual aids must be eliminated and others introduced on a wide scale.

We have given much attention to the visual principle because to a significant degree the success of a grade school teacher's work depends on a proper understanding of it. In concluding this section, all that remains to be said is that it would be more accurate to speak not of the "visual aids principle" but of the principle of "interaction of speech and visual aids" as derived from the research of Zankov and his collaborators [72, 75].

The material presented in this chapter testifies that at the basis of this principle is a definite law—the intimate interaction of concrete and abstract thinking in the learning process.

5. Developing children's skills for independently solving problems

To enable children to solve problems without assistance is one of the most important aims of primary education. In the primary grades, not only knowledge, but also skills necessary for any independent activity, must be established. These are, the ability to gain insight into a problem's meaning, the ability to analyze the problem's conditions and become aware of the question posed in it, the ability to break up a complex problem into a number of particular questions whose answers lead to the solution of the basic problem, and the ability to plan one's work and check it while solving the problem.

One condition crucial for forming these skills is the organization of a purposeful system of suitable exercises in the teaching procedure. In this regard, it is very important for children to carry out the exercises not just under the direct guidance of the teacher, but also totally unassisted.

The principal feature of a pupils' independent work is that he lacks the teacher's tutelage, which he has been accustomed to for some time, and he is left to face alone the task before him. In doing the assignment,

he should test his powers and, by resorting to his own knowledge, abilities and skills, observation, imagination and sometimes even inventiveness, find ways of solving the problem and bringing it to a satisfactory conclusion.

In connection with this, it is perfectly natural for certain mistakes to arise in the course of problem solving, for one mistake to bring about another, and for an incorrect answer to be obtained as a result. Does this mean, then, that the teacher should help the pupil at the very first difficulty? When a real threat of error appears, should measures always be taken to protect him from the possibility of error? Should the teacher interfering with the pupil's train of reason, remove any difficulty that arises and direct his thoughts and actions along the proper channel? Is it not more useful to give him a chance to ascertain for himself the error in his solution, to attempt to find his own mistake and then correct it? One cay say with certainty that educationally, the latter course will be more advisable whenever the pupil is sufficiently prepared for the appropriate work.

The teacher should interfere only when the assignment to be done independently does not correspond to the pupil's level of preparation. Then, of course, the teacher should take measure to remove certain difficulties (those beyond the child's powers!), but in so doing he is depriving the child of genuine self reliance. The above does not reduce the value of the teacher's guidance of the work. What comprises this guidance, of what is it composed?

First, the teacher thinks through the content of the assignments and the sequence in which they will be introduced, to ensure, on the one hand, the accessibility of every problem, and on the other, a gradual increase in difficulty. Moreover, all work that prepares the children for doing a certain assignment without assistance proceeds under the teacher's direction. The teacher takes care that the children have all the knowledge, skills and habits necessary to complete the given assignment by themselves. Finally, the teacher thinks out and determines the form and content of the directions accompanying a specific assignment and observes how every pupil's work is proceeding.

At all these stages, one has to consider three phenomena. First is

the peculiarities of the study material upon which the assignments (which the children are to do independently) have been built. Second is the stock of knowledge the pupils in the class have had to master earlier. The third is the individual differences in preparation, the characteristic mental processes of each child. Thus the teacher's guidance should embrace the most diverse facets of work at the preparatory stage. But the ultimate goal is to prepare the children to do an assignment totally unassisted, without direct help from the teacher in the course of the work.

With this kind of teaching procedure, the pupils' independent application of knowledge takes on great meaning. We distinguish two kinds of application. The first kind takes place when the pupil is faced with the task of learning new material (a concept or law). Here, by solving definite problems, he learns from his own experience what criteria enter into a concept and what features characterize the arithmetical rule being studied. (In presenting the problem of the principle of the activation of learning, we have already mentioned that if a student is passive, he cannot fully master the material.) In the case described above, application of knowledge is subordinate to the task of acquiring knowledge, which is one of the most important means of learning.

We are dealing with the second kind of application of knowledge when application occurs as an independent process. In this case the pupil uses the acquired knowledge to solve new school problems in different kinds of practical activity. Correct organization of both forms of application of knowledge plays a decisive role in building the pupils' ability to work independently and to think rationally.

Application ensures one very significant point that positively influences the result of instruction. It makes it possible for the teacher constantly to check the child's assimilation of the material. By giving pupils exercises in which new material is introduced, one can ascertain how well they understand it, and later, by assigning problems to which this knowledge can be applied, one can check whether the child has formed the proper skills. Regularly receiving such "feedback" from the pupil about his knowledge and skills facilitates the subsequent construction of teaching procedure, since it becomes clear to the teacher what material requires further explanation, what material needs to be repeated, and what skills need more work.

The well known, ancient proverb says, "Repetition is the mother of learning." Now it is sometimes opposed by another, "Application is the mother of learning." The latter answers better the present tasks of our school, but one should keep in mind that application includes repetition.

We do not mean monotonous, unvaried repetition, but the kind that calls for an alteration in the knowledge itself as well as of the conditions for using it.

Independent study assignments requiring pupils to apply previously acquired knowledge differ both in degree of complexity for the children and in the nature of the activity that takes place in carrying them out. Assignments based on the pupils' imitation of the teacher, reproducing his actions and reasoning, are the simplest. This work is used in arithmetic lessons in the lower grades with a view to instilling the skills of writing figures correctly in the first days of school, by doing a series of practical assignments linked, for example, with constructing a paper meter, demonstrating fractions, forming measuring skills, and so on. The substance of this kind of work is that the teacher gives an example, accompanying his actions with the necessary explanations. The children are to follow carefully his demonstration and explanation and then reproduce them when the teacher tells them to. The goal of these tasks is, as a rule, the formation of certain practical skills.

Somewhat more complex are assignments that require the children independently to apply knowledge, skills, and habits, previously acquired under the teacher's direction, to conditions analogous to those under which they were formed. By this we mean so-called practice assignments, of great significance in arithmetic instruction. Let us consider as an example the study of tabulated arithmetical operations (addition tables in first grade, multiplication tables in second grade). Since the children are supposed to learn these tables by heart through studying them, teachers conduct numerous exercises in solving a large number of the same examples from a table. Such exercises are conducted both when the teacher works with the class as a whole, and as independent work by the children in solving examples. The main idea is always the same: The children solve the same examples from the table which they had considered

previously when they were helped and guided by the teacher.

Here the child's operations are independent in the sense that throughout the assignment he is to maintain his concentration himself and strain his memory without the teacher's stimulating influence. When doubts and difficulties arise, he should resort to known means of determining the result, recall the suitable reasoning, independently find the answer, and verify it. The teacher has specially instructed the children in all these processes during the previous assignment, so that even here we are dealing only with activity that "reproduces" or "carries out," but this time this activity proceeds at a much higher level than when doing assignments based on imitation.

Assignments that require children independently to apply the knowledge, skills, and habits they acquired earlier with the teacher's guidance to conditions differing more or less from those under which they were formed, are the next, still higher level. Assignments of this kind differ fundamentally from those considered above. Depending on how much the conditions in which the knowledge is used differ from the conditions under which it was learned, this kind of assignment will demand increasingly intense independent thought and initiative. Let us show with individual examples how the degree of difficulty of these assignments can be increased.

The simplest and most widely occurring case is when the teacher, after going over a new rule with the children, gives them examples of the rule to solve by themselves (the difference between this and the above is that then we were talking about solving examples identical to those in the tables; here, about solving examples similar to those studied with the teacher, but not the same ones).

This is how fourth graders become acquainted with the rule that says, "To find the unknown dividend, multiply the divisor by the quotient." This rule is explained in a number of examples which at first are performed under the teacher's guidance. Then as independent work, the children are given examples to solve in which the letter x designates the unknown dividend [50: Ex. 720]. The corresponding exercise given in the text embraces various cases: when the divisor is a two-digit number and the quotient has a single digit; when the divisor is a three-digit number and the quotient is a two-digit number; and so on. Such a selection of

exercises introduces a certain variety into the conditions of applying the rule. However, these variations do not belong to the elements of the assignment which are relevant in applying the particular rule. Indeed, in all of these examples the same assignment form is used, which indicates to the pupil exactly what rule should be followed in solving: He reads Problem No. 720, "Find the unknown dividend ($\underline{x}$)," then he reads the example, "$\underline{x} \div 32 = 8$," recalls the proper rule, and acts in strict accordance with it.

A somewhat different picture is observed when, for example, the children are told to invent an example of dividing by a two-digit number so that the quotient comes out 307. Here the change of conditions touches not only the problem's secondary but also its basic elements, affecting the choice of method in carrying it out. (The wording is general--instead of "Find the unknown dividend" we have "Make up an example of division," familiar from working under the teacher's direction. Instead of the customary problem form, we have "$\underline{x} \div 64 = 281$" etc., determining the exact position of the unknown).

Here the pupil has to make extra effort to give meaning to the problem, to classify it under a general rule, and to apply this rule to the new conditions.

Such problems can be considered intermediate between drill assignments and creative ones--those which require the children to search for ways of solving by themselves. Creative work requires children to show self-reliance in stating a question and/or searching out ways of solving, to make the necessary observations independently, and to draw a conclusion independently.

These are assignments related to supplementing, transforming, and independently constructing examples and problems according to the varied assignments of the teacher, or freely composing them, restricted by no conditions. These are also tasks directed at independent analysis and generalization of observed facts conducted in inferring a new rule. These are assignments that require children to search independently for material (numerical material, subject material) to make their own problems.

Of particular interest are assignments that require children to seek

out and select the most effective, rational methods of solving. Let us illustrate what has been said with a few cases from classroom practice.

One of the fourth grades was told to solve the following example: $3 \times 2 \times 7 \times 2 \times 5 \times 5$. The teacher told them in advance to find the quickest way to solve it. The pupils grouped the numbers in six different ways (for example, $5 \times 5 = 25$, $25 \times 2 = 50$, $50 \times 2 = 100$, $100 \times 3 = 300$, $300 \times 7 = 2100$), but the teacher told them there was a faster way. At last, under the teacher's direction, the children arrived at the most rational grouping, $(2 \times 2) \times (5 \times 5) = 100$, $3 \times 7 = 21$, $21 \times 100 = 2100$.

Thus, by solving examples, the children formed the habit of searching for the most rational ways of solving. Such a habit should be cultivated in children, with various material, systematically and according to a plan, taking into consideration that it will not form by itself. Moreover, we frequently observe the reverse tendency in young children who take "the line of least resistance," so to speak. Without pausing to think where conditions demand it, they give the first answer that comes into their head, choosing the way they are most accustomed to.

This comes out clearly in the following case: Fourth graders were given the following example with concrete numbers: "The dividend is 1 day, the divisor is 60 minutes. Find the quotient." It was found that all the pupils (but one) in two parallel classes solved this example inefficiently. They broke up the 1 day into hours, getting 24 hours, then they broke up the 24 hours into minutes, getting 1440 minutes, then they divided this number by 60 minutes. The rational way of solving the example, which only one student used, was the following: He broke the day up into hours and converted the 60 minutes into hours, which allowed him to escape cumbersome calculations. He obtained the answer 24 quite simply by dividing 24 hours by 1 hour. This boy knew how to overcome the tendency to repeat the breaking-up operation and found the economical way of employing two opposite operations.[6]

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[6]Both illustrations stressing the importance of choosing rational ways to solve examples are taken from an experiment by the Moscow methodologist, M. S. Nakhimova.

Arithmetic problems provide very valuable material for molding skill in choosing the most effective methods. Particularly difficult, complicated problems do not always need to be used for this end. Even with simple problems one can teach children to reflect on the problem's condition, analyze it, and thereby wean them from the harmful habit (very frequently observed in primary schoolchildren) of considering calculation the chief part of the problem and therefore hurrying to get a numerical answer.

For example, a valuable exercise of this type is presented by the problem:

> There were 60 kilograms of grapes in two boxes. 16 kilograms were transferred from one box to the other. Then how many kilograms of grapes were in both boxes?

The purpose of this and similar problems is to train the pupils to refrain from a hurried numerical solution. As the data of one investigation showed, a number of third and fourth graders handled this problem successfully. After carefully acquainting themselves with the condition, they answered without making any calculations, "It's still 60 kilograms," "It doesn't change." By contrast, quite a few children followed the unreasonable course of making calculations (60 - 16 = 44, 44 + 16 = 60), without even noticing that they got the same 60 kilograms as the result.

Thus, to cultivate children's creativity in the process of instruction, and to form in them the habit of reflecting where circumstances require it, is an objective without the realization of which it is impossible to teach children effective methods of working independently. Later it will be necessary to form, step by step, the ability to study, watching so that the pupils learn to organize their study activity correctly. It is very important to teach children to use their perception for the task before them. For example, if a textbook gives a drawing or diagram having to do with a certain arithmetic problem, students should know how to use it as it is intended to be used and to see in it what can help them solve the problems. In some cases, the intended use is to see an illustration in the drawing explaining what objects are the subject of the problem. In others, the intended use is to direct one's perception to another goal, that of seeing or discovering in the diagram the basic relations given in the problem's condition. The children's approach to

these two types of drawings should be different. In the former they consider the illustration at the initial stage of acquaintance with the problem's condition and pay no more attention to it in the course of solving. In the latter, however, they use it in solving. It is fully possible to develop this ability to selectively perceive illustrations in primary schoolchildren. In an experiment in a Moscow school (No. 115), this was accomplished in the third and fourth grades. The pupils in these grades, considering the various illustrations in the textbook dealing with the problems, said, "This picture explains what a tractor is," or "This picture helps to solve the problem." In the first case they had recourse to the drawing only at the very beginning of acquaintance with the problem's condition, while in the second, they studied it extensively in the course of solving.

Similarly, pupils have to know how to organize their memory properly. The view has gained wide circulation that in primary schoolchildren, the mechanical memory predominates over logical, meaningful memory. That is, children memorize without even trying to make any sense out of the material, to establish a connection between the facts being described, or to distinguish what is most important.

This view, as the work of many psychologists shows, is not quite correct. Such a predominance does, in fact, occur, but only under certain conditions, such as when a primary teacher neglects to work at helping the children form rational methods of remembering. But if this work is conducted, primary schoolchildren can be specially taught to use methods of making sense out of the memorized material. In particular, they can be taught to turn their attention to the textbook rules and definitions in bold-face type, to check themselves when they do their homework (this is very important), and to these ends, to reproduce or say to themselves (after reading it in the book) the rule or definition to be learned. Later they can be taught not to limit themselves to examples cited in the book but to invent their own.

In teaching children effective methods of thinking, it is not enough to give them relatively new problems. It is important to bring home the basic rules of solving problems rationally. But that is a special problem, and we shall consider it in the chapter on problem solving.

6. Skill formation

We have been talking about the role of active, creative thinking in arithmetic instruction. However, active thinking is possible only if a significant number of arithmetical operations can be performed fast enough and without expending thinking effort, thus freeing time for intensive mental work on new, more complex material. Therefore a very important task in arithmetic teaching is the formation of "automatized" skills.

"Automatization" designates the transition from a conscious to a mechanical performance of an operation. Such a transition takes place in the formation of a great number of study skills—reading, writing, calculating, and others. First they are carried out consciously, and then mechanically, without the participation of the consciousness.

In case of necessity, however, when an obstacle is encountered in an operation, the consciousness again comes to the rescue and the operation is executed with its active participation.

A most important condition in forming adequate skills consists in ensuring sufficient awareness of them at the very beginning and only then bringing them into the category of automatized operations. Observance of this condition makes it possible for pupils to carry out many necessary study operations quickly and mechanically and at the same time to implement them under the control of consciousness. Then the children can always (at the first request from some source, or if they themselves feel uncertain about the answer) go back to the level of consciously performed operations.

The source of a substantial number of errors in schoolwork, particularly in calculation and computations, is premature automatization of operations. Pupils who have begun to perform calculations mechanically without sufficient awareness of them display utter helplessness if they make an error. For example, a first grader who had multiplied incorrectly from the table ($3 \times 4 = 14$) could not correct his mistake by himself, since he did not realize that multiplication is the addition of equal addends. And when we gave him visual material (sticks, matches, and so on), he could not use it to perform multiplication.

How is the teacher to proceed in these cases? First, he must not

begrudge the time spent returning such a child to past stages of instruction and re-establishing the whole chain of reasoning lying at the basis of performing the operation. To this end it is necessary to draw on visual material.

The skills formed in the process of arithmetic instruction differ by degree of complexity. Some of them represent a direct link (or association) between perception of the condition and the answer, such as the skill of multiplying from a table--after the pupil has perceived the condition, such as "7 × 8," he immediately gives the answer 56.

Another category of skills is the chain of connections, in which case the answer can be obtained only through a whole series of intermediate links, in which every link entails the next one. For example, in multiplying 158 by 200, we perform an operation consisting of three sequential links: 1) discarding the zeroes in the multiplier, 2) multiplying 158 by 2, and 3) attaching the two zeroes to the product. Some of these operations are carried out very quickly, so the zeroes can be discarded almost at the same time as multiplying 158 by 2.

At the initial stage of instruction, a series of definite rules was the basis for these operations, but when the skill of calculation has been elaborated, the pupil performs operations without recollecting the corresponding rules. These operations can thus be called "rule conforming" operations.[7]

Study activity consists of a great number of these "rule conforming" operations. Here, the rules in accordance with which operations are completed are general. That is, they hold for a large number of distinct, concrete phenomena. This can be clearly seen in the example cited above discarding zeroes and then attaching them to the product is done for any numbers multiplied.

The opportunity to rely on a general rule in developing skills is a means of very great economy in schoolwork, particularly in fulfilling a computation task. The need to memorize a large number of particular cases disappears when one relies on a general rule. If, for example, a pupil

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[7]The laws governing formation of "rule conforming" operations are revealed, studied, and minutely described by Shevarev [60].

has become aware of the general rule that multiplying numbers by one does not change them, he can multiply any number by one quickly and correctly and avoid the necessity of memorizing that $5 \times 1 = 5$, $6 \times 1 = 6$, and so on.

If the pupil has become aware of the general rule for the formation and naming of fractions, then, reasoning by analogy, he can form and name any fraction quickly and correctly, even not one yet mentioned in class.

Not only the pupil's comprehension of the rules on which they are based, but also the system of exercises he carries out, have significance for the successful execution of automatized operations.

Cases like the following are well known in school practice. Pupils learn the words to a rule adequately and can even adduce suitable examples, but they make mistakes when they apply the rule. Frequently teachers explain errors of this type by saying, "The child wasn't being careful." But why did lack of caution show up in this particular case? This in turn requires explanation. The reason for a mistake should be sought in the pupil's previous work, and in the exercises he carried out.

In every year of schooling, but especially in the first and second grade, we discover that in solving a series of examples, the children did not carry out the operation indicated by the sign. For example, they added when they were to subtract. From examining their previous work, it is easy to notice that before solving the subtraction example, they had to solve several addition examples in succession. Thus the students continued to add the numbers as if from "inertia," without noticing that the operation sign had changed.

An analogous phenomenon can also be observed in solving type-problems. If a pupil has solved several "sum and ratio" problems in a row, it may be observed that he tends to use the type method of adding the parts even for a non-type-problem if it contains a ratio. Apparently, in solving examples of the same type, a peculiar "principle of economy of mental activity" begins to operate--the children do not note those aspects of the changing reality that remain unchanged, paying attention only to the features of it that change. The pupils stopped noting the operation sign because it had remained the same (addition) in a number of examples, but they were always aware of the numbers since they had been different in every example.

However, the "economy principle" applied in these cases is false. It is necessary to pay attention to the operation sign every time and to the structure of the mathematical task as a whole. The children must know how to switch quickly from one operation to another, responding immediately to the new signal. To form these skills, examples and problems requiring different methods of operation must be alternated on a broader scale. After an addition example, give one in subtraction; after a type-problem with a ratio, assign a non-type-problem with a ratio.

The principle of variety or heterogeneity of exercises is used in practice by our country's foremost teachers. As can be seen from what has been said, it has extensive scientific substantiation.

As we have seen from the preceding exposition, varied exercises are necessary at all stages of instruction for various goals. In learning concepts and laws, they are necessary in order to separate the varying irrelevant features from the relevant ones, which are constant. In independent problem solving, they are necessary for forming thinking skills. Lastly, in carrying out automatized operations, they are necessary for developing the child's capacity to switch from one operation to another easily, overcoming the inertia of operations.

Also, a system of varied exercises in arithmetic instruction promotes growth in the activity of the child's personality.

7. The principle of full realization of children's cognitive possibilities for their age

In the preceding exposition, in characterizing the conditions of effective teaching, we had in mind primarily teaching methodology and touched on the problem of course content only in passing, when we elucidated the rule of contrasting. We said that to increase the effectiveness of learning, the study of certain similar related themes (or problems) must be brought closer together in time so that the children will be assured of an opportunity to become aware of the similarities and differences in the material being studied (and applied). In this connection, it is pertinent to decide whether topics are correctly grouped by year of instruction and within the program for each grade, or whether the sequence of topics should be changed.

But this is only one aspect of the complex problem of constructing a curriculum. At present we cannot be limited to a single regrouping of

topics--the need for a more radical change in the curricula has arisen, particularly in the arithmetic curriculum for the primary grades. This need has been brought about, above all, by the swift growth of science and technology and the necessity of including a number of new subjects in school courses. Naturally, under these conditions the question arises, is it not possible to raise the level of the demands on the children, offering them more complex material, beginning with the first year of school?

To answer this question, it was necessary to study the process by which children learn arithmetic under changed conditions that included more complex study material (compared to existent programs). This was done in a number of investigations. Zankov and his collaborators taught primary schoolchildren experimentally and succeeded in getting through the existing curriculum (in all subjects, among them arithmetic) in three years rather than four [74]. El'konin and Davydov [14] introduced elements of algebra into the elementary mathematics course in the first year of school, and Skripchenko [64] did this in the third- and fourth-grade curricula.

All researchers who had tried to teach elements of algebra in the primary grades have reached the conclusion that it is fully within the powers of primary schoolchildren. Therefore, existing programs are underestimating the cognitive possibilities of primary schoolchildren and are not using these possibilities fully.

But the legitimate question arises: Should the sole guide in constructing curricula be what children <u>can</u> learn? Children can be taught a great deal, we know, but, it may be asked, what do they <u>need</u> to study? In deciding course content, we must be guided above all by the goals facing our schools, we must take into consideration the present status of science and the system of constructing an academic subject at every grade. Only in the light of what <u>should</u> be mastered should we consider the question of what <u>can</u> be mastered.

Thus, full utilization of children's cognitive possibilities must be realized in respect to the other demands, both educational and scientific, upon course content. The authors of this book have tried to proceed from the entire sum of desiderata in working out the foundations of an

experimental curriculum for the elementary mathematics course.[8]

[8]The principles on which the curriculum is constructed are set forth in an article by Menchinskaya and Moro [40].

CHAPTER II
THE INTRODUCTION OF NUMBERS, COUNTING, AND THE ARITHMETICAL OPERATIONS

Formation of the concept of number and of the arithmetical operations is a difficult and crucial task of arithmetic instruction. The history of the development of methodology testifies that the approach to revealing these concepts largely determines the direction of all subsequent arithmetic instruction.

What is the best way to lead children to an awareness of number remains debatable even today. In this regard, it is instructive to analyze how, in the history of the development of human culture, the very concept of number originated and developed, how it was continually enriched with new content, and what significance this concept has at the present level of development of mathematics. We by no means share the opinion that the modern child should necessarily take the road traveled by preceding generations in mastering the number concept and becoming acquainted with counting and the arithmetical operations. However, we do believe it important to imagine the situations and the practical problems for which people first needed numbers, and a broadening of the original concept of number, in the course of apprehending the quantitative aspect of their environment.

Such analysis is still especially important because fundamental discrepancies in the elaboration of a methodological system aimed at forming a concept of number are explainable, we feel, by the particular fact that their authors, in concentrating on some one function of numbers, one aspect of the concept, wrongly push other aspects of it into the background and sometimes completely overlook it.

We do not intend here to give a detailed historical essay of the development of the concept of numbers. (For this question, the reader

can refer to [15]). We shall recall only the basic functions in which numbers appeared in man's practical activity, and the basic approaches to a theoretical interpretation of this concept that have been made in the development of mathematics.

1. The origin and development of numbers, counting, and the arithmetical operations

In his practical activity, man must constantly deal with the most diverse quantities of objects (the quantity of birds or beasts killed in the hunt, the quantity of fish caught, the quantity of animals tamed, and the like). The problems of stocktaking, sharing, and barter that came up in practice demanded more and more accurate means of numerical analysis of these quantities of objects so that they could be compared.

When two groups of objects at hand at a given moment had to be compared, it was enough to use the method of correlating their elements by pairs (the method that in subsequent mathematics received the denomination "establishment of one-to-one correspondence between quantities"). This method of comparing groups of objects might have been good enough if barter had always been conducted under such conditions. In practice, of course, this is far from the case. What if a fish caught today is to be exchanged for some birds the hunter has not yet obtained but will in the future? The new problem gives birth to a new method. From direct correlation of objects to be traded, man goes to indirect comparison of two quantities of objects. To this end any third quantity may be used which can replace one of the quantities being compared in the given situation.

The most successful of these quantities turned out to be the fingers and toes. So, juxtaposing one at a time the objects being transferred "in advance" and, say, the fingers of the hand, people began to use the concept of "hand" as a description of the number of traded objects. For a "hand" of fish I should get a "hand" of birds later. Thus, a man not yet proficient in counting and numbers, in the proper sense of the words, could reason in making a trade.

All right, but what if the number of bartered objects does not correspond to the number of fingers on a hand--what if it exceeds the number of

fingers and toes? A better method is devised. For example, as many scratches are made on a stick as fish traded to a hunter. Later, on receipt of the bird traded for a fish, the fisherman crosses out as many scratches as birds he has received in payment, until all the scratches are crossed out.

As social relationships developed, more and more words appeared in man's speech denoting some quantity—along with natural groups serving as a description of quantities ("how many wings a bird has," "how many fingers on the hand," etc.), certain conventional units appeared.

The ever-growing needs of practice could not always be satisfied by this method of describing a quantity. Another step forward had to be taken which would make it possible to describe any quantity--to fill in the gaps between separate word numbers. It required perhaps many centuries to make this step, but it was finally taken. The greatest achievement of human reason was subsequently to systematize the large number of word numbers--to establish a sequence such that every next word would designate the number of objects in a group containing one more object than its predecessor. The series of word numbers ordered in this way became the universal standard quantity by means of which it became possible to compare any size aggregate of objects in conditions under which the use of previously known methods would be laborious.

What did each word in this sequence mean? The word "five," for example, acted as a substitute for "hand." It reflected the property of a series of object groups (regardless of what objects composed them) that all elements of these groups could be put into one-to-one correspondence with the fingers of the hand. The word "two" reflected what is common to all pairs (be it a pair of eyes, a pair of horses, or a pair of mittens). Along with this meaning, however, every word number also gained a new, hitherto unknown meaning—each word number allowed one, when counting, to judge the place at which a certain object would be. This dual meaning of a number found reflection in man's speech--along with cardinal numbers (one, two,....) there appeared ordinal numbers (first, second, ...).

In this way, the ordering of a great quantity of numbers, in

addition to the advantages it provided in determining the capacity (number) of quantities of objects, led to a broader use of numbers and counting. The use of a number to indicate an ordinal numeral acquired great significance not only in itself, but also greatly simplified the counting of objects in a group formed from two given groups (when the number in each group is known) and determining the number in a group obtained by removing a portion from the known group.

Thus before, to determine the number in a large group left after removing one or two objects, one had to count all the remaining objects. Now it became possible to reason like this: "I had eight sheep, one I slaughtered (the eighth), the last one in the count will be the seventh, and seven sheep were left in all." Similarly, instead of recounting from one, it became possible to add another quantity to the given quantity, thus joining them. Only beginning at this point can one talk about the origin of the first arithmetical operations, addition and subtraction, since here, for the first time, numbers appeared in a form abstracted from concrete objects, and the operations were done with numbers rather than objects. This made possible the examination of numbers as such.

Man's cognition of the nature of numbers in the operations of arithmetic required tens of centuries and still has not been perfected. Moving from one relative truth to another, the science of numbers, their properties and operations using them--arithmetic--gradually exposed various aspects of these concepts by examining different applications of them in practice.

Thus the various scientific ways of treating numbers, counting, and the arithmetical operations were born. Let us consider scientific theories reflecting the two basic functions that numbers perform in practice. The first is "set theory," elaborated by one of the greatest scientists of the late 19th-early 20th century, George Cantor.

Definition of number on the basis of this theory is linked to the concept of equivalence of quantities or sets.[1] The concept of a set is

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[1]The terms podobie (similarity), ravnomoshchnost' (equivalence), and and ekvivalentnost' (equivalence of sets) are sometimes used on a level with the term ravnosil'nost' (equivalence).

Let us recall that equivalent (similar) quantities are those between whose elements a one-to-one correspondence can be established, i.e., a correspondence in which every element of the first set can be assigned to a corresponding element of the other, so that distinct elements of the first will correspond to distinct elements of the second, and vice versa.

not defined; it is used as a point of departure. No matter how the definitions of number originating from set theory differ in detail, the main feature is always the same: A natural number means what is common to all equivalent sets; a natural number is a description of the capacity (or number) of a set.[2]

The arithmetical operations of addition or subtraction are defined in Cantor's theory as the operations of combining sets or removing part of a given set. Thus, we see that in this theory number is considered as primarily a description of quantity, and the development of the idea of number, in Cantor's theory, reflects the historical path of the origin and development of numbers in human practice. That is why Cantor's theory is sometimes called the "genetic" or "cardinal number" theory.

Starting with the work of the Italian mathematician Peano, who attempted an axiomatic construction of arithmetic, still another approach to disclosing the concept of number and of the arithmetical operations has begun increasingly to penetrate the science of numbers. In this theory, a number was considered a conditional symbol conforming to the system of axioms (assumptions without proof and supported by human practice). Essentially, the properties of the set of natural numbers are formulated in the axioms. The comparison of numbers and the operations of addition and subtraction are inseparably linked with the properties of this set.

Let us give examples to show how the relative size of numbers is determined according to each of these theories. In set theory, to compare two numbers one must resort to a one-to-one correlation of sets of objects corresponding to the numbers. The number corresponding to the set of greater capacity (the one in which "extra" elements remain after pairing off its elements with elements of the other set) will be considered larger. However, to compare two numbers on the basis of the properties of the natural numbers (Peano's theory), it is only necessary to know which of these numbers comes earlier in the series. The smaller is the

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[2] Thus, in Andronov's textbook for departments of elementary school teacher-training institutes and colleges [2], this definition is given: "The general, invariable thing that completely characterizes each of a certain class of quantities is called a natural number" (all equivalent quantities form a "class of quantities").

one which comes earlier, the larger comes later.

For example, the sum of two numbers a and b, according to the axiom theory, will be a number c, located in the bth place after a in the natural number series. In contrast to set theory, which, as we mentioned above, is sometimes called the theory of cardinal numbers, the axiom theory is called the theory of ordinal numbers. We see that the latter theory deals exclusively with abstract numbers and operations with them as such, without relying on actual operations with quantities of objects. The axiom theory is a higher degree in man's cognition of numbers and arithmetical operations. It is considered more logically rigorous.

Both theories which we have considered show the dual nature of numbers and reflect the basic functions they fulfill in practice. These aspects of numbers are inseparably linked, since every word number named in counting can at the same time be regarded as the ordinal number of the last object enumerated and as the quantitative (cardinal) description of the group counted.

Everything said above has referred to the use of numbers for quantitative analysis of discrete quantities (quantities composed of separate objects). However, man constantly encounters continuous quantities (extension, weight, area, and the like) in his practical activity. A quantitative evaluation of them is no less important than counting objects. This task is performed in practice through the invention of methods of measurement.

As is known, the comparison of two homogeneous quantities, one of which fulfills the function of a "measure," comprises the basis of measurement. But measurement is inseparably linked to numbers and counting. (For example, in unrolling five meters of material, a clerk measures one piece of material after another on a meter stick, counting up to five.) The number obtained in measuring can be considered the result of counting. By the very nature of the matter, however, it acts in a new capacity here, expressing the ratio of one quantity to another.[3] The operations

[3]This new capacity in which numbers appear here leads to an extension of the number concept, since in many measurements it is impossible to express the ratio of two quantities by a natural number. When one succeeds in finding a common measure, by means of which both quantities can be measured in whole numbers, their ratio is expressed in the usual fraction form. If, however, the quantities being compared are incommensurable, the search for their ratio leads to the appearance of irrational numbers.

of counting and measuring differ quite substantially, and the meaning of the numbers obtained as a result of these operations is fundamentally different, though they remain indissolubly connected (as we observed in the comparison of cardinal and ordinal numbers).

It is useful to dwell on yet another function of number, which it acquired in the course of the development of abstract mathematical theory. We have in mind a number which functions as an "operator." Let us show by examples in what cases a number has the value of an operator. The multiplier shows how many times the multiplicand should be repeated as an addend, i.e., what arithmetical operations should be performed on this number. Analogously, in the expression a^b the number b will serve to indicate how many times the number a should be used as a factor. Thus in these and similar instances, numbers act in a new capacity. Still another area for applying them is opened up.

Simplifying the problem, we can say that in practice, cardinal numbers indicate how many separate objects are in a group; ordinal numbers indicate in which place a given object falls when counting; numbers obtained as a result of measurement indicate how many times a certain quantity is contained in another; numbers in an "operator" role indicate which arithmetical operations should be performed with another number. It is fundamental to methodology that every one of these aspects of numbers can be shown in terms of any other.

Thus we see that "number" is a multifaceted phenomenon; consideration of only one of its facets would lead inexorably to unjustified restriction of this concept and would limit the possibilities for applying it in practice. The task of arithmetic instruction consists, in this respect, in making children aware, in the most economical and rational way, of number as united in all its aspects, to acquaint them with all of its most important functions, and to teach them to use numbers, counting, measurement, and the arithmetical operations skillfully in solving various practical problems. Let us see how this task was solved and is being solved through the methods of teaching primary-grade arithmetic.

2. Basic trends in the methods of introducing numbers and the arithmetical operations

Progressive educators and methodologists have always striven to prepare mathematical theory so that the information being communicated will

be within the children's power to learn consciously and durably and yet will retain its scientific quality. In this regard, difficulties observed in children studying mathematics have been explained, mainly, by the unusual abstractness of the concepts being formed, on the one hand, and by the concreteness, the visual quality, and the pictorial nature of children's thought, on the other hand. The only possible way to overcome this contradition between the demands made on the children by the material itself and their mental capacities seemed to be maximal use of visual aids and a strictly graduated transition from the concrete to the abstract in the teaching procedure.

From the time of Pestalozzi (rightly considered the father of arithmetic teaching methodology) up to the present, these principles have determined the approach to forming children's concepts of numbers, counting, and the operations of arithmetic. It is obvious that with such an approach, there could be no possibility of constructing a school course in arithmetic according to the axiomatic scheme. The opinion of Professor Felix Klein, famous as the head of the reform movement directed toward introducing, into the school, instruction that meets the contemporary level of the development of science, is typical in this regard. He writes:

> The child will never understand if we introduce numbers axiomatically, as objects having no content and upon which we operate according to formal rules established by our own agreements. On the contrary, he unites real images with numbers; to him they are nothing other than quantities of apples, nuts, and similar good things; only in this form can these concepts be transmitted in primary teaching, only in this form will they actually be transmitted to the children [29].

This is why Cantor's set theory, which opens up the widest possibilities for the use of visual aids, should be the theoretical basis of the introduction of numbers. Practical operations with specific object groups should be the basis upon which the concept of abstract numbers, and the arithmetical operations, are formed.

All basic methods systems, as a rule, originated from such a genetic approach in arithmetic instruction. However, the sequence and interrelation in which numbers and arithmetical operations should be studied

were decided differently in these systems, and as a result, basic arithmetical concepts were revealed differently in them. All the same, without going into detail, let us examine from this viewpoint, the best-known methods systems in arithmetic instruction, which have been designated as the "number study method" and the "operation study method."

As the name indicates, the study of numbers was the basis of instruction according to the first of these methods. Comprehensively considering each number separately with the pupils, the teacher had to bring them to perceive clearly how a number is composed of addends and factors in all possible combinations. They came to know the operations of arithmetic on the basis of a sound knowledge of the structure of numbers. The consequence was the ability to perform the appropriate computations.

Children taught by this method succeeded in receiving a clear idea of number structure only through numerous exercises in which they had to consider groups of sticks, circles, and the like, corresponding to the number being studied. The arrangement of these sticks, points, and circles was to help children "notice" the composition of the number "at first glance," on the basis of direct perception.

The idea of forming a "group image" in the children corresponding to each number became especially popular after the teaching experiments of Laie, on the basis of which they were given a system of square number figures, which quite graphically illustrated the structure of each number and its link with all those preceding it.[4]

The method of studying numbers proposed by Grube, a German educator of the last century (a follower of Pestalozzi), was brought to Russia through the labors of the famous Russian educator and methodologist Evtushevskii. Evtushevskii perfected Grube's system and prepared arithmetic textbooks, good for their time, that facilitated the introduction of this method of number study into the Russian primary school.

The main flaw of this system, which later forced progressive Russian educators to speak out against it, was that children received an extremely

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[4]Laie's number figures are used in studying the first ten numbers in our schools, but only as supplementary visual material, along with various other kinds of apparatus.

one-sided mathematical development. As a matter of fact, under this system, children acquired a large fund of ideas, thanks to which all the arithmetical operations were reduced, in the last analysis, to the mental resolution of a number into its addends and factors. The arithmetical operations themselves, as mental operations with abstract numbers based on a knowledge of the properties of the natural numbers and the principles of the structure of the decimal system, receded to second place here. The number concept that was formed in the pupils became limited. Chiefly, it reflected number in its function of describing a quantity. However, an understanding of the laws governing the structure of the natural series, of oral and written numeration, of the operations of arithmetic, and of basic calculation methods was neglected.

The method of studying numbers did not provide adequate presentation of the implications of the properties of the natural series for describing each number and for methods of carrying out the arithmetical operations. It artifically impeded the children's acquaintance with basic principles of the decimal system, which allow knowledge of number properties and the arithmetical operations (a knowledge which is acquired in a smaller province of numbers), to be transferred to numbers of greater size. For example, the addition of 3 and 36 was considered only in studying the number 39, and it was assumed that the children could not make the appropriate calculations by themselves until they had studied this number, even though they had already considered 6 + 3, 16 + 3, 26 + 3, and others.

The school artifically held back the development of children's abstract thought, since the logic of mathematics was moved into the background, in comparison with the formulation of visual "numerical ideas." So it was no accident that the struggle against the monographic study of numbers led to the replacement of this teaching method by an essentially new method, emphasizing just this aspect of arithmetical knowledge--the method of studying the operations.

The transition to the operation study method was, undoubtedly, a progressive phenomenon. Teaching based upon it significantly advanced the pupils' level of theoretical preparation for further study of mathematics. However, the rush to base all instruction on children's early

understanding of the properties of the natural numbers led to the other extreme. Without a sufficient stock of life impressions, the children learned formally the knowledge communicated to them. But the abstract mathematical laws which were to guide them in carrying out certain operations with numbers sometimes lacked real meaning for them; they lacked a solid foundation of sensory perceptions.

Less and less attention was given to acquainting the children with the composition of numbers. The idea was that knowledge of the composition of numbers, in this system, had to come of itself from numerous calculations made by the children on the basis of their knowledge of the properties of the natural numbers and the laws of the arithmetical operations. So, in adding within the first ten numbers, children were obliged, for example, in finding the sum of five and three, to use the technique which they had learned of rearranging addends and then to use the method of adding units or groups. The appropriate chain of reasoning was supposed to look like this: "To add 5 and 3 is the same as adding 3 and 5; 3 consists of 1 and 2; first I add 1 and 5 and get 6; I add 2 and 6 and get 8. Then, 3 + 5 = 8." By performing such calculations many times, the child was finally to remember that 3 + 5 = 8. If he wanted to check himself when he met this example again, he had to perform it again, relying on his well-entrenched knowledge of the natural numbers and the methods of calculation that were described.

A child who had been taught by the number study method would have answered the question immediately, without reasoning it out, based simply upon his clear idea of corresponding groups of objects (if only the fingers). The second way in this particular case (which uses small numbers) is simpler and easier for children to grasp; it is more rational. But the picture changes when the operations are to be performed with large numbers. Knowing the basic methods of calculation allows the child to find the answer through reasoning. The method reduces a more difficult case to an easier one which he has studied adequately earlier. If he is not proficient with these methods, he has only his memory to rely upon--if it fails him, he must resort to visual material, without which he may be helpless.

Thus, instruction by operation study armed the pupils with a more nearly perfect means of solving by giving them general rules which they could apply on their own to any specific case. However, it sometimes caused extra difficulty for them, especially at first, since it did not take into proper account the child's peculiarities of thinking. Furthermore, with this teaching method, the idea of ordinal numbers moved into the foreground, and considerably less attention was given to showing numbers as a description of quantity.

On the whole, it can be said that the operation study method--brought forward to counterbalance the number study method, which gave a limited notion of numbers and the arithmetical operations—was itself somewhat limited. In both systems, regarding a number as the ratio of two quantities and acquainting the children with the operator function of a number were not special tasks. Numbers as ratios were considered mainly in connection with the study of fractions; numbers as operators were used in practice in the transition to multiplication, but their new function here was not brought home to the children.

The contemporary methodology of forming the concepts of number, counting, and the arithmetical operations is based on the general principles elaborated by the partisans of the operation study method. At the present stage of methodology, however, certain correctives have been included in this method which aim at eliminating the shortcomings that have been noted.

3. The methods currently used in our country to introduce numbers and the arithmetical operations

Nowadays, when the cultural level of the populace of our country is rising from year to year and the network of preschool institutions is constantly expanding, children first meet numbers, counting, and the arithmetical operations long before systematic instruction begins in school. Unfortunately, until now almost no work has been done on methods of elementary instruction of children at home (before the child starts school), although methods of teaching the fundamentals of arithmetic in kindergarten have seen quite a bit of attention in our educational literature. The system by which kindergarten curricula have been constructed is presented in full in Leushina's book, in its second printing, which

every primary schoolteacher would find very useful [35]. The system elaborated by Leushina is a combination of the number study method and the operation study method.

In accordance with the curriculum and methodology, children in kindergarten arithmetic class continually deal with quantities of objects. From perceiving groups of several objects as a whole, they gradually move on to resolve the quantity into its individual elements. They are taught how to correlate the elements of two groups of objects by pairs and how to compare size of the groups on this basis (establishing the equality or inequality of the groups being compared, and determining which of them contains more objects).

But along with this kind of exercise, the children are taught how to count objects. Having mastered the operation of counting, they use it in practical exercises to compare two groups of objects. Instead of correlating their elements one-to-one, they enumerate the elements of each group. On this basis a link is later established between adjacent numbers in a series. Thus the foundation is laid for studying the properties of the natural number series. The ordinal number is considered along with the cardinal number.

Even in kindergarten, the children get acquainted with different methods of calculation. In this connection, the study of an addition table is broken down into the three stages of consideration of the cases 1) in which the sum does not exceed five, 2) in which a smaller number is added to a larger, and 3) in which a larger number is added to a smaller.

During the first stage, children learn the appropriate cases from the table "on the basis of the study of number composition," Leushina says in her book. That is, the children learn primarily on the basis of concepts that have been synthesized as a result of numerous practical exercises in breaking down a given object group into two parts. However, the second stage of work assumes that the method of adding the second addend to the first by units has already been used. The third stage (adding a larger number to a smaller) concerns schoolwork, in which the corresponding calculations are performed by rearranging the addends.

Thus we see that the preliminary work done in kindergarten as preparation for regular arithmetic work in school envisages all around

knowledge of numbers. The children get to know numbers as a description of the size of groups of objects. They learn how to use numbers and counting to determine the relative size of groups of objects. In counting, they become familiar not only with cardinal numbers but also with ordinal numbers. Having learned the natural number sequence, they investigate how it is constructed (they ascertain that every next number is the result of adding 1 to the previous number). All of this leads them in earnest to a conscious use of the properties of the natural series in calculating, and to actual performance of arithmetical operations with abstract numbers. At the same time, during this preliminary, preparatory work, serious attention is given to having the children accumulate a large stock of concepts which help them to analyze how the numbers being studied are composed of addends. We see that the method of introducing numbers, counting, and the arithmetical operations here described combines features of both the number study method and the operation study method.

The same principles are the basis of the methodology of introducing numbers, counting, and arithmetical operations which is used in the first grade of our general education schools, a methodology which we shall now consider. Quite clearly, if children received the kind of preparation in arithmetic envisioned by the kindergarten curriculum, the work done in regular school could be planned as a natural continuation and development of preschool instruction. But mass inspection of the preparation of children entering first grade, performed repeatedly by the Academy of Pedagogical Sciences, has shown that children differ greatly from one another in this respect [9, 44, 55].

The general conclusion of careful study of the characteristic features of the knowledge, skills and habits, of seven-year-olds when they enter school is that, despite a rather large fund of such knowledge, it is still inadequate. It needs replenishing, strengthening, and systematizing. All of this work has to be effected in a class whose pupils differ sharply in arithmetic preparation. This is one of the greatest difficulties any teacher inevitably encounters from the first days of classes, assuming that he intends to continue developing the arithmetical concepts the child formed before entering school.

The current methodology solves the problem the following way. Before initiating systematic study of the arithmetical operations (addition and subtraction) up to ten, from which one could begin if all the children had the preparation specified by the kindergarten curriculum, one must conduct intensive preparatory work, directed at more precisely defining, replenishing, and systematizing the knowledge that the children have previously accumulated. This work, which is often repetitive for children who come from kindergarten well prepared, envisages a more exact definition of basic ideas and the like, numerous exercises in counting objects, and the formation of the concept of "so many" (based on counting). Thereafter, the children proceed to study the first ten numbers in order. Usually two lessons are allotted for each number, and they are conducted, as a rule, in a specified manner. In the first lesson the children learn how to form a new number by joining one to the previous number (by means of corresponding object groups and practical operations with them). They then study how to form a group of objects corresponding to the new number, and practice counting up to the number being studied (in both directions, using concrete visual aids and square number figures). Once familiar with the characteristic features of the printed and written numeral corresponding to the number being considered, the children do exercises in correlating the numeral with the quantity and vice versa, as well as exercises which require them to know the place of the numbers studied in the series (naming the number's "neighbors" in the series, naming the one following the given number or the one preceding it, remembering a gap in the series, and the like). The lesson as a rule ends with the children writing the numerals in their notebooks.

The focus of the second lesson in studying a number shifts to an analysis of how it is composed of addends. Every possible combination of two numbers adding up to the given number is considered through actually dividing a group of objects into parts, and by working according to illustrations in the textbook and number tables. Along with object demonstrations, square number figures are used in studying the composition of numbers.

As they are studying numbers, the children are meeting their first arithmetic problems and examples. At this level of instruction, they

solve only such problems and examples which require a unit to be added to a given number or taken away from it, and they determine the result on the basis of practical operations—joining one object to a given group of objects, or removing one object from a group.

After all ten numbers are examined this way, one after another, the teacher proceeds to introduce addition and subtraction up to ten. The method proposed by the textbook is that, from this moment on, children should not perform addition and subtraction by counting, but rather by adding on (and taking away) units and then groups (twos, threes). The order in which all cases are considered is: "Add and take away one," "Add and take away two," "Add and take away three," and so on. After adding four, children are visually introduced to the commutative property of addition, and use it later in adding a larger number to a smaller. Learning all the cases in the addition and subtraction tables by heart should be the result of numerous exercises in adding and subtracting by units and groups.

This methodological system of working on the first ten numbers has gained wide popularity in our schools, especially since it was made the basis of the standard first-grade arithmetic text. However, a number of criticisms were leveled at it in the works of Soviet methodologists. These remarks usually dealt with the order of introducing the children to individual questions of this topic, and ended with a specification of the content of work at separate stages of studying the topic [1, 10, 57].

In certain methodological investigations, it has been noted that in actual school teaching, insufficient attention is given to creating in children the sensory foundations necessary for forming the concept of abstract numbers, for comparing numbers, and for studying the properties of the natural numbers, and that in this connection some knowledge of arithmetic in children is formal from the very start.

Let us demonstrate with some examples. To explain the concepts of "bigger" and "smaller," separate objects (a big ball and a smaller one, etc.) rather than groups of objects are used for comparison in the first lessons. In such a demonstration, the pupils do not become sufficiently aware of the arithmetical meaning of these concepts. Later, when they study the first ten numbers and are asked which of two numbers is larger

and which is smaller (for example, 5 or 4), the children orient themselves by a formal feature (the number's place in the series) when they answer, without being able to check their conclusion experimentally (by comparing corresponding groups of objects). The teacher is not verifying the children's grasp of the arithmetical meaning of these basic concepts. He is not guiding the formation of these concepts without which conscious mastery of numbers and the operations of arithmetic cannot be ensured.

Another criticism that deserves attention concerns the way in which children learn the order of the natural numbers. Numerous observations, as well as the special experiments cited [9, 37, 44, 56] prove that children learn the order of the first numbers of the series easily (as a rule, before starting school), but their corresponding comprehension is almost inert. Though they easily reproduce this order in the forward direction beginning with 1 (1, 2, 3, 4,...), they experience considerable difficulty when told to continue a series starting with any other number (4, 5, 6, 7, ...), to count backward, or to name the number following a given number or preceding it. Also, knowing how to count from one allows the pupil only to perform the task of counting objects. For all further work connected with the study of arithmetical operations and number properties, confident and flexible mastery of the series is necessary.

To ensure just such mastery of the number sequence, it is necessary to organize suitable exercises. Observations show, however, that many teachers depend excessively upon frequent repetition of their number sequence in the forward direction, "from the beginning" (starting with 1), and fail to pay enough attention to exercises directed at more flexible mastery of the number series.

Specific ways of using the various schoolroom visual aids to form children's concepts of numbers and the operations of arithmetic have also been subject to criticism. It has been mentioned that the method of using visual aids should vary according to the tasks confronting the teacher at each separate stage of forming these concepts.

A failure to meet this obvious requirement can be seen, for example, in the following case. Having introduced the first ten numbers, the teacher moves on to consider addition and subtraction. The chief tasks at this period of instruction is to ensure the children's advance from

the method of counting from one to the method of adding (or taking away) by ones and groups. If the children then are given only concrete visual aids (i.e., objects) in solving examples and problems, as they did earlier, the instructor not only fails to make the transition easier, but on the contrary makes it more difficult for them to advance to the new method of determining the result. Indeed, if, for example, the remainder which the children are to determine in solving a subtraction problem is presented in visual form, it is easier for them to count the corresponding objects than to use the subtraction method, which is less rational under these conditions.

Clearly, when a new problem is put forward at a given stage of instruction, the way in which visual aids are used should vary so that the teacher creates conditions for completing the assignment which set off the advantages of the new method of solving, or better yet, which absolutely require its application. Various partial uses of concrete visual aids (e.g., in which one of the addends is given in number form and the other is illustrated with an object) suit this purpose (for more details, see [42, 56]).

The above and other criticisms of the method of forming concepts of numbers and the arithmetical operations are, we see, directed toward a more exact definition of particular questions, toward perfecting the present system, and do not generally concern its substance.

The merit of the method used in school to form the most important concepts of arithmetic was not subject to doubt until very recently. Everyone agreed that the concept of numbers and arithmetical operations was to be formed by giving the children abundant experience in actually working with quantities of objects. In the course of comparing groups of objects, establishing one-to-one correlations between elements of compared groups, joining groups of objects, etc., the concept of numbers as descriptions of quantity, the concept of numbers as ordinal names for objects being counted, and then the concept of arithmetical operations with numbers, were to take shape in the children. No one was confused by the circumstance that enlargement of the number concept--acquaintance with other functions of numbers--was deferred for a rather long period, and when the task of broadening the number concept arose in the course of

instruction, children often met with substantial difficulty, since it was then necessary to rearrange previously formed knowledge, skills, and habits. Such phenomena have been observed in introducing fractions, negative numbers, and so on.

No objections were ever raised, either, against the general approach taken in acquainting children with the principles of mathematics, an approach which was charted by the curriculum and can thus be defined in very simplified, schematic form: from practical operations with real objects (mainly before primary school and during the first weeks of school) to operations with numbers and the study of their properties (the first 5 or 6 years of school); from considering the properties of numbers and the four arithmetical operations with specific examples and from considering the interdependence of certain specific quantities to discovering general properties of numbers and the operations of arithmetic expressed in letter symbols and to studying various forms of interrelation between quantities in generalized form (from 7th grade on). A gradual transition from the concrete to the abstract--from consideration of particular cases to generalization (for example, from operations with numbers to letter expressions), gradual enlargement of the number concept by the introduction of certain new number forms and subsequent generalization of the numbers studied (by comparing them with ones considered earlier)--this is the primary trend in the formation and development of basic mathematical concepts which has been determined and reinforced by experience.

In recent years, however, this general trend in arithmetic teaching and above all, the very approach to forming the concept of numbers, considered to be tested and confirmed by centuries of school instruction the world over, has been cast in doubt in the works of certain investigators. The urge, so charcteristic of our time, to revise radically the entire system of teaching mathematics in school (particularly in the lower grades) is explained primarily by the fact that against the background of grandiose successes in science and technology in modern times, against the background of rapid development of the exact sciences, the inadequacy of the mathematical preparation of the graduates of our schools is being felt more and more.

The relatively insignificant improvements and corrections that have been made in our curriculum and methods up to now still have not been able to ensure elimination of the gap between what the school gives the students in this regard and the requirements of life. The creative thinking of methodologists and psychologists, scholar-mathematicians and practicing teachers is involved in a search for a way to overcome this contradiction. To everyone interested in the organization of mathematical education in the school of the future, the projects for radical revision of elementary school arithmetic instruction advanced by various authors and whole scientific groups today, both in our country and beyond her borders, cannot fail to arouse the liveliest interest. So let us look in somewhat more detail at certain modern trends in elementary school mathematics instruction that have direct bearing upon the problem of introducing numbers and the arithmetical operations.

4. New trends in methods of introducing the principles of mathematics

In this section, certain new trends in methods will be considered, all of which call for radical reorganization of primary school mathematics teaching. One of them is linked to the name of the Swiss educator Cuisenaire. The "new method of teaching arithmetic in primary school" that he worked out has been widely disseminated in the schools of a number of foreign countries.

Cuisenaire posits as the basic task of arithmetic instruction the pupils' conscious mastery of mathematical relationships, based on practical operations with special teaching material. To this end he proposes the use of rods of various sizes and colors that would replace definite numbers. All the operations of arithmetic and the correlations between numbers are revealed, in his system, with the aid of these rods.[5] For example, in introducing the concept of multiplication as the addition of equal addends, the author uses one long orange rod and five short red ones whose aggregate equals it in length. The appropriate entry is made: "1 orange = 5 red," or "5 red = 1 orange."

[5]For a detailed description of Cuisenaire's devices, see [7].

Cuisenaire similarly shows the relationship between the different operations of arithmetic: three light-green rods and one white one are matched in length to the orange rod, so that the pupils perceive various possible corrections visually. For example: "3 light green + 1 white = 1 orange," or "1 orange - 3 light-green = 1 white."

However, this is all determined by color (without numbers). In various practical exercises, for example, a rod three units long is replaced by three rods (1 unit) or two bars (2 + 1). All this looks very unusual to our eyes, but this method can facilitate comprehension of certain mathematical relationships because it is quite graphic. In this case, the pupil has the opportunity to perceive visually a number of combinations that show diverse relations between quantities and between the arithmetical operations.

From the study of mathematical relationships through color, Cuisenaire leads the pupils to study them "in terms of the numbers from one to ten." The expression "in terms of the numbers" is not accidental here. For Cuisenaire, numbers are not the object of study, just as, earlier, color was not. Both are only a means to another end--the comprehension of mathematical relationships.

Characteristically then, the value of using "semi-abstract" material that can replace (symbolize) mathematical relations in teaching arithmetic, lies, according to Cuisenaire, in the very fact that it is "semi-abstract." He criticizes bringing everyday situations into arithmetic teaching. He notes that objects from the environment should not be objects of calculation in arithmetic lessons, since they distract children from discovering numerical relationships.

School instruction in our country aims to effect close links with life. As we see it, in contrast with Cuisenaire, the use of numbers in everyday situations not only does not lead the pupil away from the basic academic task but, quite the contrary, it facilitates more conscious learning of mathematics, an understanding of its link with the practical. From associating with adults, a child begins very early to choose objects to count in his environment. By the time he comes to school he has already accumulated his own arithmetical experience from his life. And no colored rods symbolizing abstract mathematical relationships can "tear him away" or isolate him from this life experience.

Indeed, even during the first days, months, and years of school, children will continually have to solve problems in life, problems requiring the use of numbers, counting, and the arithmetical operations. To deny for an extended time any guidance whatsoever to children in solving these everyday problems, to create an exceedingly artificial situation in which they have to perform various manipulations with "semi-abstract" material in mathematics classes (and they are unable to comprehend either the sense or ultimate aim of the manipulations) would mean to repudiate the leading principles of instruction and education upon which the work of the Soviet school is based.

Cuisenaire's approach to numbers also raises fundamental objections. Keeping in mind that one of the most important tasks of the school is to nourish a scientific, Marxist world-view in children, it is exceedingly important to construct mathematics instruction from the very beginning so as to make them aware of the link between this science and practice. Children should realize that the number concept has been adopted by man from the real world and is not a product of pure thought, and therefore, number should be a special object of study in school.

However, these fundamental objections to Cuisenaire's initial positions do not preclude the use of individual methods he has recommended in practical instruction in our schools; in particular, one should try using rods of various sizes and colors to show the pupils visually the varied relationships between numbers and the arithmetical operations, as well as basic arithmetical laws. The arithmetic counting box of rods, which we use, is not sufficiently adapted to these ends.

Individual attempts by Soviet psychologists to revise radically the content of mathematics education, particularly its elementary stage, deserve more attention. Thus, Gal'perin and Georgiev's work, in which they elucidate the results of experimental kindergarten teaching, regards as paramount, in the formation of the concept of number, the analysis of the numerical relationships between quantities through measurement. In one of their reports we read: "The mathematical approach to the quantitative aspect of things begins with the isolation of a 'measure'" [23]. Therefore, in their next report, which discloses a program for instilling elementary mathematical concepts, they chose the "formation of a mathematical approach to studying quantities" as the first topic [22].

The authors see one of the chief tasks of the study of this topic to be to teach the children not to estimate size by the impression they get from a direct comparison of dimensions, but to systematically use a ruler and the results of measurement.

All work involved in studying this topic, which includes formation of the concepts "larger," "smaller," and "equal," is done without numbers or counting. When two homogeneous quantities are compared with the help of a chosen common measure, chips (or notches, etc.) are used instead of counting in the comparison. Every time the measure can be put on one of the rods (or one glass is measured off, etc.) one chip is set aside; the groups of chips so obtained are compared by establishing a one-to-one correspondence, and a conclusion is drawn--which of the quantities compared is larger, which is smaller--according to the results of the comparison.[6]

Twenty-two lessons are allotted for such preparatory work, the purpose of which is to establish a link between a quantitative estimate of quantity, and its measurement with a unit of measure. Then the children proceed to study numbers (38 lessons). At first they form the concept of unit, formulated by the authors thus: "That which is measured and equals a unit of measure is unity, or one." All subsequent numbers are formed from 1 by adding 1 and are studied consecutively according to this plan: forming a new number and naming it, applying the number in measurement, cardinal and ordinal counting, addition and subtraction up to the given number.

After studying numbers, special time is allotted to considering, in generalized form, the relationship between a quantity, a unit of measure, and a number. It is explained that one cannot compare numbers obtained from measuring heterogeneous quantities, or from measuring homogeneous

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[6]The third report [22] relates in detail how the experimenters begin with exercises in which the children compare quantities composed of similar objects, chosen with the aid of a unit of measure (there is a unit for every kind of object). The direct comparison is replaced by indirect (with the help of chips, marks, etc.). Finally the exercises create a situation in which numbers are necessary, in connection with tasks of the form: "Go into another room and get as many... as there are things (chips) here--but you may not take the things with you." Hence the advance to counting and using numbers as a description of quantity.

quantities, or from measuring homogeneous quantities using different units of measure. At this stage, a series of practical exercises is conducted showing how a number obtained from measuring depends upon the unit of measure by which measurement of the given quantity is accomplished.[7] And the reverse situations are also considered--how the unit of measure must vary if a number is assigned and the quantity increases or decreases, and how the quantity must change if a number is given and the unit of measure increases or decreases.

We can see that the system of introducing numbers, counting, and the arithmetical operations proposed by Gal'perin and Georgiev [22] has much in common with the one currently employed. Indeed, numbers even here are introduced by actually establishing one-to-one correspondences between the elements of groups of objects. Fundamental to the introduction of numbers here is the concept of "so many" that arises from comparing object groups.

The chief difference between the way children are introduced to numbers now and the one contemplated by Gal'perin and Georgiev is that in the latter, work begins by comparing continuous quantities introducing the principles of measurement and doing suitable exercises without using numbers or counting (until they are introduced in class). According to the experimenters, children should first perceive numbers as the result of measuring (as the ratio of the quantity being measured and the chosen unit of measure). However, counting and numbers are actually introduced in exercises to replace the chips that had registered how many times a unit was contained in a measured quantity. In this connection, in fact, numbers are introduced, as usual, to describe quantitatively a group of objects in this experiment (by establishing a one-to-one correspondence between the numbers arranged in natural sequence and the chips making up the group being counted).

Actually, as we have seen, children meet numbers for the first time in exercises that require an answer to the question "how many?" (that is, exactly how many separate objects--chips, notches, and so on--not "how many times is the unit of measure contained in the quantity being measured?"). True, familiarity with numbers and counting is used

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[7] See preceding footnote.

immediately in connection with measurement. So when children are taught experimentally in Gal'perin and Georgiev's system, they are introduced earlier than usual to numbers obtained not only as a result of counting but also as a result of measurement. They meet numbers not only as a description of the quantity of individual objects making up the group being counted (numbers as "a collection of units"), but also as indicators of ratio.

From the very beginning of instruction, the important fact is brought home to the children that a number depends on the unit of measure chosen, that the unit of measure is part of the quantity being measured but is by no means identical with the concept of unity in itself (even discrete quantities can be counted in pairs, in threes, etc.). Such an approach to forming the number concept is of undoubted interest.

In criticizing the modern teaching method, Gal'perin and Georgiev stress that it does not give sufficient attention to numbers as an indication of the ratio of one quantity to another. Galanin pointed this out in his time, when he wrote more than 50 years ago:

> ...Elementary instruction insistently builds the idea that a number is a collection of homogeneous counting units, but this idea certainly does not contain the idea of ratio, although...the number concept is, rather, contained in ratio, for which a collection of counting units is a particular instance [21].

One must admit, apparently, that this defect in the method of numbers study has been preserved to the present day. Gal'perin and Georgiev have outlined one possible way to overcome this defect.

Then there is doubt about the advisability of planning instruction so that, in arithmetic class, for a relatively prolonged period of time, the children do not use what they have learned about number. It is also very doubtful whether lessons devoted to introducing numbers should create a conflict, to a certain extent, between what the teacher says and the knowledge the children have acquired earlier. The question arises, can the defects of present methods mentioned above be avoided, at the same time guaranteeing, in the first stages of instruction, a natural continuation, development, and extension of the relatively rich store

of knowledge, skills, and habits in operating with numbers and counting which children learn before school.[8]

The same questions arise in considering the experiment conducted under the direction of Davydov. This experiment proposes a radical revision of the entire system of teaching mathematics principles. Davydov gives much attention in his research to questions connected with the study of numbers and counting [13].

Davydov's basic plan is to begin teaching mathematics in school by developing in children an understanding of the most common mathematical relationships, stated in letter symbols. He proposes that basic numerical relationships between quantities be considered first so that numbers, their properties, and operations with them will appear as particular cases of already familiar general laws.

Accordingly, Davydov allots three-fourths of the first year of school to considering the basic relationships between quantities--forming and comparing the concepts of "larger," "smaller," and "equal"; considering the conditions of maintaining and violating an equality, going from an inequality to an equality and vice versa; considering the relationship between elements of an equality. All the while the children have nothing to do with numbers: they operate only with specially selected teaching material (rods, containers of water, sand and scales, etc.).

In the study of equality--how it is maintained and violated--addition and subtraction signs are introduced. The children learn to express as a formula the transition from equality to inequality and the reverse. For example, $A = B$, $A - e > B$, $A > B - e$; or $A > B$, $A - e = B$, $A = B - e$.

The experimental instruction showed that first graders learn these relationships in letter symbols without numbers and before numbers. Numbers and counting are introduced, in Davydov's program only in the second semester. Numbers are introduced as the ratio of two quantities, which are compared by measuring them.

After solving problems related to measuring various quantities (of length, weight, etc.) with the aid of various measures, the children become familiar with this kind of task, for example: "Put the number 3

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[8] Gal'perin and Georgiev [22] investigated kindergarten teaching, where these questions are less meaningful than for work in the first grade.

on the axis, taking 4 boxes for a unit (a measure)," and so on. In this way they become familiar with the principle on which the natural numbers are constructed. Relationships between quantity, measure, and number are then considered. Exercises of this form are carried out: "Quantity A by measure g equals 6; the same quantity A by measure e equals 8. Which measure is larger, g or e?"

The next part of the program calls for introducing addition and subtraction of numbers, but unfortunately, how these arithmetical operations are inculcated is not considered in detail in the published materials, which only say that addition and subtraction are introduced "by means of reducing an inequality to an equality." Examples like this are cited:

$$\begin{array}{r} 8 > 5 \\ 8 - x = 5 \\ 8 = 5 + x \\ \hline x = 3 \end{array} \qquad \begin{array}{r} 3 < 7 \\ 3 + x = 7 \\ \hline x = 7 - 3 \end{array}$$

It is explained that in both cases x is found first by using a table (the instructor asks how much should be added to 5 to get 8, and the children look at the table and find 3). The table is then memorized by means of the exercises.

Furthermore, it can be seen from the published curriculum that later, numbers will be understood as the ratio of the quantity being considered to the quantity that is taken as a unit of measure, a notion employed particularly in explaining the meaning of the operations of multiplication and division.

Neither the procedure or teaching experimentally according to the curriculum under discussion, not its results, have yet received sufficient elucidation in print. So it is not possible to make a detailed analysis of the merits of the proposals. We must, however, turn our attention to the most controversial aspects of Davydov's proposed system, and methods, of forming the number concept and of teaching children counting and the arithmetical operations. It was mentioned above that Davydov's experiment raises the same doubts as did the view of Gal'perin and Georgiev. In

connection with Davydov's proposals, however, we must raise a few additional questions of fundamental importance.

The first question: Can one arm students with the skill of "immediately manipulating quantitative features in generalized form" on the basis of practical work with sticks, paper ribbons, blocks, and plasticine, with water, weight, and other such teaching material? Will this generalization be so broad (as the experimenter is convinced) as to make it possible later to consider corresponding questions with numbers as a particular case? This question cannot be answered merely on the basis of theoretical analysis. It needs to be studied in actual instruction. What is more, even if suitable research is conducted which answers the question in the affirmative, it is still necessary to check whether the method contemplated meets the criterion of raising the effectiveness of instruction as a whole.

The second question: Is it right to use letter symbols at such an early stage of instruction for writing the generalization being taught, as Davydov proposes? There is every reason to fear that such early introduction of letters can lead to the unwarranted "formalization" of instruction and to its isolation from life. The fact that children, judging by the data cited in Davydov's article, learned to carry out exercises linked to the use of letters in mastering the described curriculum does not prove that they will be able to supply the knowledge acquired to practical problems.

Finally, the third question that should be contemplated when considering this system is this. By placing the idea of ratio in a position of prime importance in teaching the number concept, studying mainly the comparison and measurement of continuous quantities over an entire year of instruction, do not the experimenters reduce the significance of a number as the result of counting, as a quantitative characteristic of sets composed of separate objects (sometimes not uniform in many respects)?

It is difficult to say definitely which tasks are more often encountered in life--those requiring measurement or those requiring the counting of separate objects.[9] Each of these tasks has its own specific character, and if Davydov is right when he reproaches modern methodology for neglecting the task of instilling the concept of number as ratio, is not he making the same mistake in regard to establishing the numerical quality of discrete quantities?

In considering the different functions numbers perform in the solution of various practical problems, we have already mentioned that each of them, generally speaking, can be seen in terms of another.

Methodologically, however, the aspect of numbers which is "turned" toward the children when they first meet them is scarcely an indifferent matter. It is also important for all aspects of numbers to be sufficiently illuminated in later instruction. The need for a system of suitable work, and for a proper sequence in considering all problems related to the children's formation of the number concept and mastery of the arithmetical operations, should be fulfilled by beginning with the general didactic and psychological principles described in the first chapter.

From this standpoint, it is more rational to lead the children away from forming the number concept on the basis of practical operations, with concrete quantities and sets of objects, to an even deeper study of the abstract properties of numbers and operations with them. This course makes it possible to organize school instruction as the natural continuation and development of pre-school instruction, to use more fully experience in operating with object groups and initial knowledge of numbers and counting, acquired by the children before they came to school, and allows

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[9] Theoretically, of course, one can consider any problem of the second type a particular case of the first. But is not this too hard for a first grader? Does he understand, for example, that when he answers the question, "How many chairs are in the room?" he has used the concept of "chair" as a measure just as in measuring water he used a glass as a measure. Is not the reverse reasoning simpler, in which measuring is considered a particular case of counting (let us count how many units of measure are contained in this quantity being measured)?

mathematics teaching to be closely related to life from the start. Under these conditions, the knowledge children acquire in the first grade allows them to solve a wide variety of tasks that arise in their playtime and school activity and in daily life. At the same time, the study of numbers and the operations of arithmetic provides a foundation for later generalizations and for moving on to letter symbols.

The problem of perfecting numbers and the operations of arithmetic and of removing the defects that were justly pointed out in the works considered above should be considered from this particular standpoint. We are attempting a solution in our experimental curriculum which is based on these considerations. This curriculum is presently being checked by the Institute of General and Polytechnical Education of the RSFSR Academy of Pedagogical Sciences.[10]

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[10]The experiment was begun in the 1963-1964 school year in Moscow Schools No. 314 and 44, in the V. I. Lenin School, in Kursk School No. 7 and Kardymovskaya School in the Smolensk province; in 1964-1965 it covered more than 40 classes in various provinces of the RSFSR.

CHAPTER III

INSTRUCTION IN MENTAL AND WRITTEN CALCULATION

1. The role of teaching mental and written calculation in an elementary mathematics course

It is generally agreed that along with the inculcation of basic mathematical concepts, the study of number properties and the laws of the arithmetical operations, the formation of children's calculation skills occupies a most important position in elementary school teaching. At various times and in various countries, fundamental discrepancies have been observed in determining the relative importance of mental and written methods of calculation, as well as in the very approach to forming the appropriate skills.

The American school, for example, has always characteristically been enchanted with written methods, and in forming calculation skills, it has always attached fundamental importance to mechanical practice exercises, which occupy most of the time allotted to arithmetic instruction.

By contrast, the Russian school has always been distinguished by its great attention to mental calculation and its desire for children to gain a thorough understanding of the calculation methods used in carrying out the arithmetical operations, in both mental and written form.

In the American school, children are taught to write addition and subtraction in "columns" while they are still studying the first ten numbers, and then, as they become familiar with these operations up to 20, 100, and over, they are taught to use written methods of calculation. Thus even in the first year of school, written calculation predominates. Our children use exclusively mental methods during the first two years of school (when the first 100 numbers are studied). Later, the children must mentally solve simple problems in addition, subtraction, multiplication, and division with large numbers, resorting to written calculation only when actually necessary.

Obligatory daily exercises in calculating mentally, performed at literally every arithmetic lesson in the elementary grades, have become a deep-rooted tradition of school instruction.

To what can we attribute the adherence to mental calculation so characteristic of our schools. Apart from the practical significance of the ability to perform simple calculations "in the head" rapidly and correctly, Soviet methodologists have always considered mental calculation one of the best means of deepening the theoretical knowledge which children acquire in arithmetic lessons. Mental calculation facilitates the formation of basic mathematical concepts, a more thorough knowledge of how numbers are composed of addends and factors, a better assimilation of the laws of the arithmetical operations, and so on. Also, great educational importance has always been attached to exercises in mental calculation. It has been felt to facilitate the development of children's resourcefulness, quick-wittedness, attention, memory, activity, speed, versatility, and independence of thought.

Comparing mental and written methods of calculation, progressive pre-revolutionary Russian methodologists often noted that in all these respects, mental calculation opens up significantly wider possibilities than written calculation. In their efforts to underline this basic distinction, methodologists of the past sometimes even exaggerated the significance of mental calculation, incorrectly contrasting it to written. For example, the Russian mathematician and methodologist, Goldenberg wrote that mental calculation deals with numbers while written calculation deals with figures. The first is natural while the second is artificial. Further, mental calculation is creative but written calculation is constrained. Authors of methods manuals opposed such a sharp juxtaposition, even in the pre-revolutionary period. They particularly emphasized that even written calculation requires a certain degree of awareness.

It is ncessary to stress another aspect as well, to which Goncharov [24] addressed his attention. To a certain degree, automatism should play a part even in mental calculation. Even then one must talk about developing particular skills on the basis of specially selected exercises. Even the stage of mental calculation that appears the most creative, that

is, analyzing the numbers subject to calculation and choosing the most rational course of calculation, requires special skills acquired only through exercise. In recent years, voices have begun to be heard concerning the unjustified "dominance" of calculations in the elementary arithmetic course [13]. They refer to the existence of handbooks of tables that children must know how to use.

What place should mental and written calculation occupy in a modern school? What more particular questions related to this topic are fundamental and require special testing in school? Both forms of calculation should retain an important role in the system of elementary school mathematics instruction. A book of tables cannot embrace all situations that the pupils will meet in life. If they carry out arithmetic transformations by themselves, it will help them understand the very essence of the transformation, without which it will be hard for them to understand even the tables. But the teaching of mental and written calculation should be streamlined by introducing expedient changes in both curriculum and methods.

The possibility of an earlier introduction of methods of written calculation up to a hundred (not up to a thousand, as stipulated by the current curriculum) should be determined later. One should also check whether concentric cycles might be curtailed by discarding one thousand as a special cycle. There are serious grounds for considering the possibility of reducing certain instances of written calculation in the curriculum which deal with multidigit numbers. Certain cases of multiplication and division by a three-digit number could be subject to curtailment first. Certainly, calculation methods are the same for larger numbers as they are for smaller numbers. The student masters these methods completely by dealing with smaller numbers. And if in practice he finds it necessary to compute with larger numbers, he will be fully able to deal with them. He knows the methods and has only to concentrate throughout the entire calculation process.

It should be noted in passing that children in school make the most mistakes in written calculations (with large numbers) not because they do not know the calculation methods, but because they fail to maintain their attention during the whole calculation process (for details, see the following section). Having the students generalize the study material

more often should play a decisive role in streamlining the teaching of calculation technique.

2. Heavy reliance on generalizing in teaching calculation

The basic methods of mental and written calculations which children must master in elementary school are based on the properties of numbers in the decimal system and on the properties of the arithmetical operations. Children are introduced to methods of calculation, however, long before they know the general laws on which they are based.

We have already talked about the difficulties which the child meets when first introduced to numbers and about his great need to consider a sufficient number of concrete facts—practical operations with quantities of objects. Even here, however, literally from the first steps of instruction, the numbers themselves should become the children's object of observation. Thus, when telling the children how the number 2 is formed by joining 1 to 1, and then how 3 is formed by joining 1 and 2, 4 by joining 1 to 3, and so on, the teacher should impress upon them the generalization that every next number can be obtained by joining 1 to the preceding one. Analogously, in considering the composition of each separate number in order, it is necessary that the children grasp the main principle--that every number can be composed either of individual units or of various groups.

Quite clearly, at this stage of instruction one cannot expect the pupils to verbalize these general principles, but one can and should stimulate them to carry over the knowledge they have acquired in studying certain numbers to other classroom material.

If these initial generalizations are reached in the study of the first ten numbers, it will be possible to raise the level of the study of addition and subtraction within 10 on that foundation. But in doing so one must consider each case of addition and subtraction separately, relying primarily on the visual methods principle, as is most often done in practice. Another way, however, is much more effective for impressing upon the children the general proposition on which these cases are based: "Every number can be added (and subtracted) in parts, either in units or in groups." The children come to this conclusion by observing the teacher add on 3 circles, 4 squares, and the like. The children do

not formulate the conclusion, but they themselves begin to act by analogy in considering new situations. Because of this, one may raise significantly the share of work done independently by the children in analyzing new material.

Here--though this procedure is not stipulated by the curriculum--it is useful to introduce the commutative property of a sum (as the textbook urges). The children are introduced to this property visually; the law is not put into words. They are required only to learn how to use it consciously and in the right place in calculations. Thus, solving the example 2 + 7 = , the pupil will say: "Adding 7 and 2 is the same as adding 2 and 7-- you get 9," and will write the answer, 9. That is enough.

In studying the topic, "The Second Ten Numbers," the children master basic methods of mental and written calculation (representing a number as the sum of a row of ones, methods of adding and subtracting with and without "carrying"). Therefore, it is very important to point out to them these general principles as well as to have them memorize the table of addition up to 20. Knowledge of these principles not only will help the children to consciously use a particular calculation method, but it will be good preparation for a later consideration of the properties of the arithmetical operations.

One can say approximately the same thing about multiplication in first grade. Here it is even more important to understand the principles and not just learn particular cases. A basic task in the first year of school is to show the meaning of the operation, to indicate the link between multiplication and addition. All work on this topic should be subordinate to instilling the concept of multiplication as the addition of equal addends. The exercises given to the children should show that multiplication always can be replaced by addition. It is best to do this with simple problems in multiplication, with the aid of an illustration of the conditions of such problems. On the other hand, it is important to show that not all addition can be replaced by multiplication, but only addition of equal addends. To this end, in addition to examples of the type: 3 + 3 + 3 + 3 + 3 = ___; 4 + 4 + 4 = ___, etc., and the assignment to replace addition by multiplication, one should give examples

in which such a substitution may not be made, as well as mixed examples: 3 + 3 + 3 + 3 + 5 = ___.

Furthermore, when studying the arithmetical operations in first grade, children can learn certain particular cases in general form, such as, for example, subtraction in which the remainder has a zero or multiplication by 1.

Under the heading "A Hundred," work continues in forming and perfecting skills of mental calculation. One can rely on what the children were taught in first grade more fully than has been done until now in studying operations within 100 in second grade. Everything should be done to make the children see how much there is in common between the new examples and the ones they solved in first grade. For this, one should first compare examples solved analogously, such as the following:

14 + 3 = ?	24 + 3 = ?	36 + 2 = ?
14 = 10 + 4	24 = 20 + 4	36 = 30 + 6
4 + 3 = 7	4 + 3 = 7	6 + 2 = 8
10 + 7 = 17	20 + 7 = 27	30 + 8 = 38
14 + 3 = 17	24 + 3 = 27	36 + 2 = 38

In demonstrating the solving method with visual aids, it is better to use the same aids as in first grade, to give oral explanations in the same form, and so on.

Children perceive the resemblance between adding (and subtracting) within 20 and within 100 fairly easily. But they do not always perceive the distinction so well as the similarity, and thus they make mistakes in solving. Consequently, it is useful to organize exercises so that examples of this kind are contrasted.

In order to study multiplication and division up to 100, the children must not only learn the proper tables, but must also explain the relation between the operations under discussion, and apply knowledge of this relationship to deduce the appropriate tables.

In second grade, children become acquainted in practice with certain new properties of the arithmetical operations. Besides the commutative and associative properties, they use the distributive property of multiplication over addition and others extensively in calculation. The corresponding laws usually are not stated here. But in fact the children, in solving appropriate examples, can already perform operations based on a

general rule, without resorting to visual material every time.

First- and second-grade experience in applying the properties of the arithmetical operations and the decimal system when studying mental calculation is an excellent foundation for the final fourth-grade drill in the corresponding generalizations.

In teaching written calculation in third and fourth grade, it is extremely important for the children to realize the meaning of the operations carried out in each specific case. Although they give much attention to teaching the children conscious methods of mental and written calculation, teachers do not always fully use, in later work, the total knowledge which the children acquire.

Thus in advancing from one concentric cycle to another, from the study of operations with numbers up to 1000 to the study of operations in the whole field of numbers, the teacher arbitrarily spends much time on explanations which add very little to what has been said earlier.

Much attention should be given to making children aware of the features of the numbers of the decimal system as they advance from one concentric cycle to another. On this basis, having noted the similarity between operations with different sized numbers, children can independently carry over their knowledge, skills, and habits (acquired, for example, in studying addition and subtraction up to 1000) when performing these operations with numbers of any size.[1]

In speaking about the role of generalization in teaching operations of calculation one must touch upon the question of introducing third graders to fractions, although this leads us slightly beyond the bounds of the topic at hand.

Fractions are introduced in third grade, according to the curriculum. However, as mentioned in the explanatory note to the curriculum, familiarity with them begins essentially in second grade, where the students are given the concept of a part of a number, they learn how to find a half, a third, a fourth, etc., of a number. In third grade this work must be developed, of course. And here a special cycle is introduced, the content

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[1] The methodological literature throws much light on the question of mental and written calculation [49: 54].

of which is listed in this way: "1/2; 1/4; 1/8; 1/10. Expanding and reducing, reading and writing fractions. Solving problems by finding a fraction of a number" [58].

Here one cannot but note that along with a natural expansion and deepening of the work in second grade, occurs a relative restriction. In fact, in second grade the children learned to find any part of a number (within the limits of the numbers and operations studied), and in third grade their work is limited only to the four fractions indicated in the curriculum and, accordingly, to fractions with denominators of 2, 4, 8, and 10.

Thus, even though the second grader has no difficulty finding a third or a fourth of a number, here for some reason it is proposed, after teaching the child to find, say, 3/4, 5/8, 7/10, to teach him to find 2/3 of a number, prematurely (?!).

Obviously, this reflects the influence of tradition, which proposes monographic study of certain specific fractions, whose number and form did not remain constant as the curricula changed. The authors of previous curricula quietly rejected the possibility of children's realizing the general principle of forming and naming a fraction even in fourth grade. In this respect, the current curriculum takes a step forward, since it proposes that in fourth grade, in solving problems involving finding a fraction, the third-grade skills and knowledge for fractions with denominators 2, 4, 8, and 10 will be expanded to finding fractions with denominators from 1 to 10.

However, in third grade the monographic quality is retained in studying parts. The experience of the best teachers testifies that after learning how to form, read, and write fractions by three or four examples, the children are able correctly to explain how to form, read, and write other fractions not yet considered in class. The students are fully able to learn the law that is the basis for obtaining any part, and to make the necessary generalizations. But in order to do this it is important not to isolate the study of one part from another. On the contrary, the teacher must show the connection between the cases being considered. If the teacher tries to instill suitable generalizations, it will make it easier for the children to study the curricular material and will make them more aware of what they are learning.

3. Calculation errors and how to overcome them

The basic idea advanced by psychological theory concerning errors comes down to this. A mistake is not just the absence of the right answers. It is the result of a definite process, the nature of which must be unmasked. This proposition is often neglected in teaching practice. Interest is shown only in the examples or problems in which the children make mistakes, and the same conclusion is made in all such cases. A mistake means additional exercise is needed. We shall not apply this conclusion to all cases.

The distinctive nature of a mistake dictates distinctive methods of attacking it. In a single example there can be mistakes of different origin. Even the same mistake, moreover, may have a varying nature. To understand how diverse errors can be, we must be guided by a definite classification. A study of the mistakes our pupils make in calculating makes it possible to draw up such a classification.

We divide all mistakes in calculation into two basic categories, according to the source of the mistake--in the conditions of carrying out a given operation or in the quality of mastery of arithmetical knowledge and skill.

Mistakes caused by conditions under which an operation is carried out are "mechanical." They arise when, because of certain conditions (fatigue, loss of interest, excitement, distraction, and the like), a child's conscious control weakens in solving examples. These mistakes do not imply lack of knowledge, nor do they imply that some arithmetical operation has not been learned thoroughly enough--they are linked only to the pupil's weakened attention as he carries out the operation. Therefore, mistakes of this kind fluctuate. An example solved incorrectly the first time is solved correctly the second time, even if the first mistake is never corrected. Among these mechanical mistakes are slips of the tongue and pen, when instead of one number, another number bearing some resemblance to it is uttered or written. This resemblance can be caused by various factors such as the sound of the number being pronounced or the way the figures are written. It may have an acoustical, optical, or motor basis.

Replacement of one idea by a similar one may occur not only in the final step of the process--writing or pronouncing the finished result--

but also at earlier stages, when perceiving the numbers (by ear or eye). This category of mistakes also includes so called "perseverative" mistakes, in which some number is fixedly retained in the mind. Such is the nature of the mistake 43 + 7 = 70; in this case the digit in the second addend was mistakenly written in the total. In certain cases this phenomenon is caused by assimilating or likening one number with another, such as in 6 + 7 = 12, in which the second addend obviously was likened to the first (6 + 6). The mistake 47 + 9 = 66 has an analogous origin. In this case the tens digit is likened to the units digit.

These mechanical mistakes are quite varied and yield to explanation only with difficulty. Therefore, attempts to explain the conditions under which this kind of mistake is more likely to appear deserve great attention.

Research has shown that in the performance of all four operations, examples containing identical digits give the greatest number of errors. In a number of cases, the "perseverative" tendency was strengthened in the presence of identical digits, which encourage errors such as $4 \times 4 = 24$ or $6 \times 6 = 66$. This phenomenon needs additional research and analysis.

Weakening of conscious control because of fatique is peculiar to written calculation. An increase in error is observed when progressing from the lower place values to the higher. This phenomenon was specially studied by Pchelko. It was found that of 200 third graders who solved the following addition example, 40 made errors--no errors in the units place, 5 errors in the tens place, 14 in the hundreds place, and 13 in the

$$\begin{array}{r} 470{,}824 \\ 15{,}782 \\ 6{,}594 \\ +\quad 7{,}986 \\ \hline \end{array}$$

thousands. In this example, adding the units, the sum of which was 16, was the easiest for the pupils; next in difficulty was adding the tens, which totaled 27, and the most difficult was adding the hundreds, which totaled 29. One might infer that the degree of error in adding the numbers in a given place-value depends on the magnitude of the addends.

To check this inference, the pupils were told to solve an example in which the sum of the units was the largest number, the sum of the tens was a little less, the sum of the hundreds still less, and so on. The

inference was not confirmed. Adding the units, the pupils made 4 mistakes, the tens, 8 mistakes, the hundreds 11 mistakes. (Thousands and higher place-values have fewer addends in the example, so that they do not enter

$$\begin{array}{r} 968{,}469 \\ 77{,}698 \\ 4{,}577 \\ +\quad 348 \\ \hline \end{array}$$

into the comparison.)

On the basis of a number of these examples, Pchelko draws the following conclusion:

> In adding several addends, the number of mistakes increases with progression to higher place-values. Obviously, a great many numbers and an abundance of operations on them quickly tire the pupils and dissipate their attention. Furthermore, every error in the preceding places influences the results of adding the numbers in the next places [47: 53].

This cause--weakened conscious control--is the basis of a very pervasive error, in which pupils fail to carry out the operation indicated by the sign. The peculiar character of the numbers given in the example may determine the choice of operation in this case. Certain numbers create very favorable conditions for a particular operation--the combination of 56 and 6, for example, creates conditions for easy subtraction. In this case, through weakened conscious control, subtraction may be performed in spite of another sign in the example.

Performing an operation which does not correspond to the sign may also be conditioned by the influence of preceding operations, a phenomenon which might be called inertia of action. For example, a pupil solves several addition examples in succession. After a while he stops paying attention to the sign and adds, even though an example has some new signs, such as subtraction. This kind of mistake is typical and is encountered at various levels of instruction.

A mistake such as $80 \div 15 = 6$ occupies a rather special place among mistakes caused by weakened conscious control. We found this error in the test papers of a number of third graders. Even adults make this kind of mistake. They usually explain it by referring to their experience--"Usually it can be divided without a remainder." The number 6 was chosen as quotient because multiplying it by 5 gives a number ending in zero.

The same process is the basis for this error among schoolchildren. Weakened conscious control in solving an example is doubtless another basic reason for this mistake. But faulty learning of division with a remainder is also involved. That is why such a mistake can be considered a transition to the next group of mistakes.

Mistakes in the second group are caused by insufficient mastery of arithmetic skills. This group can be divided into two large subgroups, according to the nature of the mental process upon which the knowledge and skills are based.

If the calculation skill is based on rote memorization of definite numerical results and if it is poorly established, the incorrect answer may vary and sometimes may even alternate with the correct answer. For example, one pupil was observed to give three different answers for the example 7×8. One time he answered incorrectly--54, another time correctly--56, and the third time, on a test, again incorrectly: $7 \times 8 = 58$.

Mistakes of the second subgroup, contrasted with mistakes of the earlier kind, relate to skills based on a general rule. The nature of the mistake is determined here by how the rule was learned, by the extent to which the rule used to perform the operation has been generalized. Such mistakes are relatively constant.

For example, on the tests written by pupils (two second- and two third-grade classes) in one Moscow school, more than 50 mistakes were made in solving the example $1000 \div 200$, but all of them could be reduced to two forms--50 or 500. In certain children we also discovered a substantial persistence of the error. On three tests separated by intervals of time, one little girl wrote the same incorrect answer. A number of pupils repeated their mistakes on two tests separated by an interval of 3 1/2 months.

Cases like this belong to the same category: A pupil who got 10 when he divided 96 by 6 left the mistake uncorrected when he checked his work, and when he checked a classmate's work, he replaced the other's incorrect solution ($96 \div 16 = 9$) with his own incorrect solution. A number of pupils retained the wrong answer of 10 in this example over rather extended intervals of time.

On what processes are these mistakes based? How does the pupil modify the known rule when he carries out a wrong operation? A study of the mistakes permits an answer to this question. A rule the pupils learned earlier acquires an unjustly broad application, or certain necessary steps in it are omitted, and as a result the rule is carried over to cases where it does not apply. Such is the origin of mistakes like 1000 : 200 = 500 or 1000 : 200 = 50. In performing addition and subtraction on numbers containing zeros, the pupils operated according to a definite rule. They performed the operation by discarding the zeros beforehand and then attaching zeros to the result. Passing on to division examples, they proceeded in exactly the same way. Thus, in the example 1000 : 200, they obtained 5 by dividing 10 by 2. To the 5 they attached a number of zeros (one or two).

The next mistake, 96 : 16 = 10, arises analogously. A pupil himself explains the steps of his calculation as 90 : 10 = 9, 6 : 6 = 1, 9 + 1 = 10. Again a rule applicable to written addition and subtraction (in which the operations are performed separately in regard to the tens and units) is carried over into division.

In the two examples cited above, the rule's application was extended to involve not only addition and subtraction, but any operation.

Mistakes engendered by the similarity of two rules are a special variety of mistakes of the second subgroup (mistakes based on general rules). There are, for example, two similar rules for short addition, and for subtraction, of 9. In both cases, one must perform the appropriate operation first with ten and the reverse operation with one. Thus one is subtracted in the first case and added in the second. In a number of cases, the pupils are not aware of this latter, particular, and differentiated task. They are aware of it only in general, indefinite form (some operation with 1 has to be performed). In this case, an incorrect answer may have even a better chance of arising than a correct one, since the pupils have a tendency to carry out the same operation with one that they carried out first with 10. For example, some pupils got 26 when they added 15 and 9, since they added 10 to 15 and added on 1 (instead of subtracting 1).

A number of researchers distinguish <u>errors conditioned by habit</u> as a special group of mistakes. However, our analysis of mistakes shows that

the concept of habit as applied to mistakes needs to be analyzed. Habit can be expressed through psychologically different processes. First, it can be expressed by the establishment of a habitual operation. In this case, it is easily overcome by exerting attention (see the examples in the first group of mistakes described). Second, it can appear in the form of a habitual generalization (mistakes of the second subgroup, just described). In the latter case, it is not enough for the child merely to exert his attention to overcome the mistake it is necessary to show him the error of the generalization and to instill new information, and then new habits.

The methods of attacking mistakes should be varied, and the choice of method is determined by the nature of the mistake. The causes of mechanical mistakes lie outside arithmetical knowledge and skills. Therefore an attack on these mistakes should be directed toward increasing the child's interest in arithmetic exercises, mobilizing his attention, heightening his sense of responsibility, etc.

The methods of attacking the other two kinds of mistakes are quite different. Here the cause of the mistakes is revealed in arithmetical knowledge and skills. Therefore the correction of a mistake must primarily concern just those skills and knowledge.

The problem is often put this way: Is there any positive value for overcoming an error by analyzing it? Or can concentrating on an error only lead to harm, so that practice in correctly solving appropriate examples must be the basic measure?

As our research has shown, it is not legitimate to make such a statement about the relative value of methods without considering the nature of the mistakes. In one case analysis of a mistake plays a very large role in overcoming it, but in another it is totally useless. When we deal with mistakes based on a misunderstanding of a rule, it is very important to analyze the mistake itself and show the pupil how it arose, so that he may realize his mistake. On the other hand, it is totally useless (if not harmful) to rivet the pupil's attention to a mistake that arose as a result of an insufficiently established skill. The pupil cannot extract any instructive lesson from analyzing the solution of $7 \times 8 = 54$. Here the sole method of attacking the mistake is supplementary exercise in a poorly established skill.

In one school, various methods of attacking errors were tested. The results showed that analysis of errors based on a false understanding of rules had a very great effect. The pupils who previously had made this kind of mistake did not repeat it. Moreover, error analysis prevented other pupils, to a significant degree, from making these mistakes later. The effect of awareness of the mistake was so powerful that the pupils who made the error remembered it. They received a kind of "intellectual burn" and subsequently were wary of the mistake. For the other category of mistakes, which resulted from insufficient retention of a skill, extra practice was the only effective method.

In attacking mistakes, much attention should be given to "prophylaxis" or mistake prevention. Every teacher knows that it is necessary above all to guarantee the pupils' effective understanding of schoolwork in order to protect them from later mistakes. But there is still another way of preventing mistakes, the significance of which some teachers underestimate. We mean the way in which exercises are selected. Even with a sufficiently thorough understanding of a rule, a child may err because the exercises using this rule were too monotonous.

These mistakes demonstrate the following law, which is characteristic of mental activity (we described it in Chapter I, but let us recall it): When homogeneous operations are repeated under identical conditions, the elements of the situation that remained constant for the duration of the repetition cease to be perceived. The law manifests itself not only in mathematical skills but in all other fields as well. When pupils solve many examples in succession using the same operation, they stop paying attention to the sign. But if one observes the principle of heterogenity in selecting exercises, the cause of the mistakes will be eliminated. The pupil who is solving examples in various operations alternately will pay attention to the sign each time. Here it is necessary constantly to make the children watch for possible changes in the condition of the examples being solved. Instead of the general warning, "Be careful," special directions can be given from time to time: "Watch out for the operation signs," "Check whether you may or may not use the rule in this case," and the like. There is every reason to believe that the shortcomings which still occur in our pupils' practical calculations occur

not so much because they do not know the rule, as because the system of exercises for actually using these rules is insufficiently organized.

CHAPTER IV
TEACHING PROBLEM SOLVING

1. Arithmetic problems in the elementary mathematics course

The question of the place of arithmetic problems in the system of mathematics education demands serious consideration.

On the one hand, the experience of the foremost educators and methodologists shows that solving arithmetic problems (using a proper method of instruction) plays a very important, positive role in mathematics instruction. On the other hand, more and more voices are being heard regarding the inexpedience of keeping arithmetic problems in the curricula, or at least regarding the impropriety of using them to the extent to which they have been used up to now.

Academicians S. L. Sobolev, A. L. Mints, and others declare that mathematics is taught in the schools contrary to "the rules of optimal strategy," so that children who are taught arithmetic must waste their energy "learning abstract thinking all over again in algebraic form." From this standpoint, arithmetic problems receive the most criticism. According to Sobolev, after mastering algebra, a child is no longer able, as a rule, to solve an earlier problem by arithmetic methods. Why then deceive the children? Why not teach them abstract thinking in the lowest grades?

Because of this controversy it is necessary, in expounding the foundations of the method of teaching the solution of arithmetic problems, first to recount clearly the real role of arithmetic problems in arithmetic instruction, why they are needed and what mental processes they develop in the children. In addition, it is very important to understand how the algebraic method of solving differs psychologically from the arithmetical one and what mental processes are carried out in each case.

Problem solving has always been considered in an arithmetic course as a study activity with a double aim. First, it facilitates mastery of

mathematical concepts and laws by showing their relationship to life, and promoting their use. Second, it has an independent value as it tends to serve to develop the pupils' creative thinking. If you analyze the arithmetic texts for the primary grades (the ones by Pchelko and Polyak, or others), you will quickly note that a substantial number of problems serve to instill mathematical concepts and laws (the concepts of difference and ratio, the operation properties, etc.). Other problems are aimed at teaching pupils particular methods of solving. These are the so called "type-problems." In addition we find, in the third- and fourth-grade texts, a section of "interesting" problems, consisting of extremely heterogeneous problems such as somewhat more difficult type-problems, examples in story form, and riddles.

To discover the role of problems in mathematics education and to analyze the arithmetical and algebraic techniques of solving them from a psychological standpoint, it is necessary to analyze different kinds of problems. A study of the problem-solving process discloses the following: In all cases the children must perform two basic mental operations--analysis and abstraction. The pupil has to analyze the condition of the problem into the data and the unknown and choose an appropriate arithmetical operation. He must abstract from the subject of the problem and translate into mathematical language, so to speak, the relationships between the data and the unknown which are described in the text in everyday terms.

When he advances to problems in two operations, the pupil's zone of choice is broadened. He must choose not only the operation but also a pair of numbers from several given in the condition. If, however, the composite problem has only two numerical data, he is required to provide an intermediate datum from the condition, and here again the choice of the intermediate datum is determined by what must be found in the problem. It is determined by the question which the problem raises.

Let us investigate the processes effected in using the algebraic method of solution, which presupposes setting up an equation in these elementary cases. First, it is necessary to determine the possible methods (and their psychological meaning) which form a transition from arithmetical solution to solution using an equation. Setting up a numerical formula (not after solving and obtaining a result, but before, when

the unknown has not yet been determined) is such a transitional form.

Let us consider as an example how a formula is used in two single-operation problems. In the direct problem:

> There were 7 apples, of which 4 were eaten. How many apples were left?

the unknown is the number of apples left, and the relation between the data and the unknown can be expressed by the equation $\underline{x} = 7 - 4$, or (if a question mark is used instead of $\underline{x}$) $7 - 4 = ?$

In the reverse problem, expressed indirectly:

> There were several apples. Four were eaten and 3 were left. How many apples were there at first?

the unknown is the initial number of apples, and the relationship is expressed by the equation $\underline{x} = 4 + 3$.

Comparing the arithmetical method of solving with the one which approaches the algebraic, we must pay attention to the following circumstances. At first glance, the use of $\underline{x}$ introduces nothing new and specific into the problem-solving process, and in the right-hand side of the equation, we are dealing essentially with a numerical formula. But there is one essential psychological point that should be considered here. When we tell the pupils to write an equation before solving, they are faced with a special task: finding the unknown, designating by $\underline{x}$ (or a question mark) what is unknown and has to be found in the problem. This unknown (in the first problem, "How many apples were left"—in the second, "How many apples were there?") determines the numerical formula (in the first case, $7 - 4$; in the second, $4 + 3$). Determining the unknown thus becomes the central, special task.

Psychologically it is also very important that the pupil be required not simply to repeat some part of the condition. Indeed, pupils do this constantly in answer to the teacher's question, "What does the problem ask?" but not uncommonly, this turns into an empty formal repetition of the final question, which does not make the pupils aware of the unknown as distinguished from the available data. The psychological factors involved in setting up an equation is something else. Here one may not limit oneself to repeating part of the text. One needs to reflect upon

the whole text of the problem, breaking it down into the unknown and the data, and to write down the result of this reflection in expanded form on the left and right sides of the equation.

The use of x has another important advantage in this case: By directing the problem solver to look for the most general feature--"this is not known," "this has to be found," it should hasten the process of abstracting from the nonessential story aspect of the condition.

At the same time it should be stressed that setting up such an equation (even in the simplest form) presupposes that the pupils are able to read the condition carefully, to understand the meaning of the concepts used, and to represent clearly the situation described in the problem. And this ability is acquired when using the arithmetical method of solving problems. So in this respect, arithmetic is necessary for algebra.

A significant number of arithmetic problems in two or more operations can be solved, like one operation problems, purely by arithmetic. In this case, however, writing down x brings something new into the solving process. Let us show this by the example of solving a two-operation problem in first grade:

> There are 6 liters of milk in one milk can and in another can 2 liters more than in the first. How much milk is there in both milk cans?

Children solving this problem often make the mistake of fusing two operations into one. They add 2 liters and 6 liters, and when they get the result, they feel they have answered the final question of the problem. What good is it to write the equation $x = 6 + (6 + 2)$? Here again the unknown, or what is sought, is separated from what is given, and the necessity of writing out the data should help in identifying the two operations--both the intermediate and the final one.

Setting up an equation gives the pupil a chance to survey the whole system of interrelationships expressed in the problem's condition, which he does before he solves the problem with numbers. When a problem is solved by arithmetic, without first setting up an equation, the pupil often is so absorbed in solving a particular problem and concentrates so hard on the calculations that he overlooks the mathematical

structure of the problem as a whole and the system of interrelationships connected with it. Solving equations of the types considered does not yet require mastery of special algebraic apparatus, since operations with the unknown are unnecessary.

Among the problems cited above was a problem expressed in indirect form:

> There were several apples. Four were eaten and 3 were left. How many apples were there at first?

To solve it, we devised the equation $x = 4 + 3$. One can, however, set it up differently: $x - 4 = 3$. And this equation, though it calls for an operation with the unknown, can be solved without using algebraic apparatus--by knowing the relation between the components of the operation (in this case, that the minuend equals the subtrahend plus the difference).

An equation for a division problem can be constructed and solved analogously:

> Twelve meters of cloth were used to make all the dresses, 3 meters for each dress. How many dresses were made in all?

In this case x denotes the number of dresses, and the equation has the form $3x = 12$. Here again the equation can easily be solved on the basis of what is learned in the arithmetic course--one of the factors is equal to the product divided by the other factor.

It should be noted that solving problems with equations which call for an operation with the unknown would not seem to be any simpler than solving by arithmetic. On the surface, the reverse is true. In solving, for example, division problems by the arithmetic method, the reasoning consists of one step, and accordingly, one operation--division--is performed immediately, whereas in composing an equation, the reasoning consists of two steps: first the operation on the unknown--multiplication--is designated, and only then is the division written out and performed.

But from the purely quantitative aspect of solving, the arithmetic method of solving is somewhat more difficult psychologically, since it assumes qualitatively original ways of thinking which differ for every problem. In this case, for example, it is necessary to imagine that as many dresses will be made as the number of times 3 meters is contained

in 12 meters, and when solving the indirect "apple" problem it is necessary to choose the operation of addition, which contradicts the terms in the text of the problem ("were eaten," "were left") which are closely linked with subtraction in the child's experience. By contrast, solving a problem by an equation supposes a method of reasoning characterized by greater uniformity and varying little for various problems. The unknown is singled out, its quantitative relationship with other data is determined, and so on. The course of constructing an equation in solving an indirect problem reflects the course of the everyday actions described in the problem:

> If x is the initial number of apples, and 3 apples were eaten, then 3 must be taken from x.

The qualitative diversity of arithmetical solving methods and the uniformity of the algebraic are even more striking in solving type-problems. In them, setting up an equation helps significantly to simplify the solving process. This proposition is true primarily for problems which require an arbitrary assumption, an artificial transformation of the condition. Among others, sum-and-ratio problems, data-equalizing problems, and substitution problems belong to this category. For example:

> Ten books and 20 notebooks were bought for 2 rubles. A book is 8 times as expensive as a notebook. How much each do a book and notebook cost?

To solve this "substitution" problem by arithmetic, we have to resort to an artificial method, supposing that only notebooks were bought, and accordingly concluding that 80 notebooks can be bought in place of 10 books (naturally, the children find this train of thought difficult and cannot do it by themselves). Then we find the total number of notebooks (80 + 20 = 100), then the price of each notebook (2 rubles ÷ 100), and so on.

The algebraic method of solving does not require any artificial assumption. Real correlations are reflected in the process of devising an equation: "The cost of one notebook is x. Then the cost of one book is $8x$." It is quite easy to devise the equation

$$(8x)10 + 20x = 200.^{*}$$

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*There are 100 kopeks to a ruble (Trans.).

One can assume that solving such an equation will be fully within the abilities of primary schoolchildren, since it is based on the knowledge of arithmetic they acquired earlier.

$$80x + 20x = 200$$
$$100x = 200$$
$$x = 200 : 100$$
$$x = 2 \text{ (kopeks)}.$$

Among the interesting problems there are quite a few which are solved much more easily with an equation.

We have been considering problems in which x has always designated the final unknown. There is, however, a category of problems in which it is expedient to solve by taking for x not the unknown expressed in the problem's final question but one of the intermediate ones. Let us give an example of such a problem:

> On the first day, 84 kilograms of cake were sold. On the second day, 192 kilograms were sold, and 8 rubles, 64 kopeks more was made on them than on the first day. How much money in all was made on the cake in the two days?

In this case, designating the final unknown by x would greatly complicate the solution of the problem. Analysis of the condition make it possible to choose the cost of one kilogram of cake as the unknown to be designated by x. Then it is easy to construct an equation:

$$192x - 84x = 864, \text{ therefore}$$
$$108x = 864$$
$$x = 864 : 108;$$
$$x = 8$$

But a legitimate question arises: To what degree is it expedient to resort to an equation in this case? Indeed, when a pupil chose the cost of one kilogram as an intermediate unknown by analyzing the problem's condition, he had already in fact solved the problem arithmetically. The arithmetical solution of this problem is made much easier when the condition is written schematically:

I -- 84
II — 192 -- 8 rubles 64 kopeks > than in I.

With such a schematic notation, the pupil can survey all at once the entire system of interrelationships contained in the condition (as he did in writing an equation), which provides a basis for choosing an operation to solve the first particular problem by looking ahead and realizing that later the cost difference is mentioned, and therefore he must not add the number of cakes sold the first and second days but must find their difference. This operation is necessary so that the cost of one kilogram may be determined. Thus a prevision of subsequent operations occurs here. This is a higher form of the analysis which plays a most important role in solving any creative problem. Enrichment of the thought process by this form of analysis is essential in describing the solution of arithmetic problems and may serve as a criterion for judging their psychological and educational value.

Apart from problems for which the arithmetical method has a certain advantage over the algebraic (one such problem was just discussed), there are others to which the algebraic method is generally inapplicable (true, there are few of them in the arithmetic workbooks). These are problems which have to be solved by comparing numbers:

> Twenty-five birches and 32 poplars grew in a schoolyard. In the fall 10 more birches were planted. Then which trees were there more of, and how many more? (Problem No. 358 in the second-grade text by Pchelko and Polyak.)[1]

So what conclusions can be drawn from our comparative analysis of the process of solving problems by two methods--by arithmetic and with the aid of an equation? All this material shows that both problem-solving methods essentially supplement each other. They reach the same goal but in different ways. The algebraic way is not always the shortest and most economical. The expediency of using one method or the other depends on the nature of the problem.

In certain cases, both methods are approximately equivalent and can achieve the goal with the same effectiveness. In other cases, one turns out to be more effective, and the advantage might be either on the side

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[1]For the rest of this chapter we refer to the standard arithmetic textbooks published in 1964.

of the equation method (as people usually think) or on the side of the arithmetical method of solving. The latter usually occurs in problems in which it is more useful to designate an intermediate unknown by x than the last unknown expressed by the problem's question. Then it is necessary to carry out arithmetical transformations, and to a large extent, constructing equations no longer makes sense.

We should isolate three basic categories of arithmetic problems according to the type of equation which can be set up to solve them. The first kind of problem calls for an equation on the left side of which is x, the unknown, and on the right a numerical formula. This equation is easily solved by a purely arithmetical method. The second kind of problem results in an equation which calls for operations with the unknown, but here the equation can be solved arithmetically by using one's knowledge of the relationship between the components of the operations. Finally, the third kind of problem also calls for equations requiring operations with the unknown, but here the equations can be solved only if the student has mastered the special apparatus of algebra (reducing like terms, etc.).

The isolation of these three categories testifies to the existence of various transitional forms of study activity between the arithmetical and algebraic. There is every reason to believe that these transitional forms should occupy a very prominent position in primary mathematics education. It should be stressed that some of them have already been used in practice in our schools, although still on a limited scale.

The game of "Guess the Number" is employed as one form of mental calculation as early as the first year of school and is essentially nothing but an elementary form of using x, the only difference being that an empty square takes the place of the algebraic symbol.

In the later grades, x is introduced in solving examples (see Pchelko and Polyak's third-and fourth-grade texts). With this in mind, pay special attention to page 91 of the third-grade book, where examples with x are first introduced, and the pupils are asked what is known and what is unknown in the examples. Afterwards, they are told to compose and solve a problem, using the relationship between three quantities: 1) the price of one meter, 2) the number of meters, and 3) the cost. The problem

cites numerical values for two quantities, designating the third (the unknown) by x (accordingly, three possible ways of making up the problem are suggested). This exercise leads directly to solving problems with an equation. Construction of numerical formulas for a problem already solved, or the reverse (construction of problems on the basis of a specified numerical formula), plays an analogous role. These exercises have a definite place in arithmetic instruction in our schools, although various teachers treat them differently and introduce them in different grades.[2]

Apparently it is necessary to include forms of working on an arithmetic problem which call for constructing equations of various types and degrees of complexity (before the problem is solved with numbers). One should take into consideration the data cited above demonstrating the positive role which equation-construction can play in solving problems that vary in difficulty--direct and indirect, one-operation or two-operation problems--and the more difficult type-problems. As was shown, the analytic processes (and the first stage of analysis, breakdown of the condition into the unknown and the givens, or data) and abstraction from the story and calculative aspects of the problem are processes which are facilitated by constructing equations.

The advisability of using equations which pupils can solve by relying only on their knowledge of arithmetic can scarcely be doubted. In this case, the proper system of introducing this material, as well as the nature and number of suitable exercises, etc., are subject to verification in teaching practice. It may prove useful to solve problems of one kind by arithmetic and problems of another kind with an equation. Obviously it will often be useful to solve the same problem first by arithmetic, and then by algebra (or the other way around) to give the children a chance to compare the methods. Furthermore, such a comparison will help to establish the meaning of the mathematical transformations and the

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[2] A teacher, N. I. Kozheurova, succeeded in having the children write a problem's solution as a numerical formula as early as first grade [31].

arithmetical operations which they perform to find the value of the unknown after setting up the equation.

A different problem concerns using equations whose solutions are based on special knowledge of algebra in the primary grades. It is necessary, first, to ascertain the feasibility of introducing this knowledge in the primary grades, since it would hardly be advisable to tell the pupils to construct equations which they could not solve. Then the use of equations in solving problems would lose its basic meaning for the pupils and would be totally unjustified. The pupils would be deprived of the chance to compare the arithmetical and algebraic methods of solving, to check the solution, and to appreciate the greater rationality of the algebraic method in solving a number of problems.

From all that has been said, one cannot agree with the current tendency to undercut arithmetic and to feel that it detracts from the basic goals of mathematics education. In considering the arithmetical and algebraic methods of problem solving, we were able to show that arithmetic plays a very important role by teaching children to perform analysis, synthesis, and abstraction. These are processes which are necessary in mastering any branch of mathematics. Arithmetic problems provide an opportunity for pupils for practice in grasping a mathematical relationship and in using a number of mathematical laws (for example, those determining how the components and the results of an operation are related); practice which is essential for constructing equations.

Above we cited the view that solving by arithmetic should not be included in the curriculum because it is difficult and can be immediately replaced by the simpler algebraic method. However, we cannot agree for two reasons. First, for certain categories of problems, the arithmetical method is no more difficult than the algebraic. Second, the very difficulty or "problematic" nature of the problem may even have a positive didactic use. That is, difficult arithmetic problems which make substantial demands on the basic thought processes develop the pupils' thinking. This view of the problem has been very clearly crystallized in Soviet methodology and is shared by many progressive teachers. "We mathematicians think nothing of giving pupils assignments requiring them to overcome difficulties, beginning in the lowest grades," writes Pokrovskaya. She continues, "Work that calls for no effort from the pupil

has no educational value [51]." The positive role played by problems of increased difficulty is elucidated by Shor [61] and Skatkin [63], who have carried out special research. The latter, together with a teacher, N. O. Zharova, experimented with difficult problems in the fourth grade. Although only a separate group of students solved them, not only they, but the whole class became more interested in arithmetic.

Many have written about the interest thus aroused in intellectual effort and in independent conquest of difficulties including both the methodologist Polyak and Honored Schoolteacher of the RSFSR, V. K. Monastyreva of Vologda. The feelings which arise in connection with solving hard problems were well communicated by one fourth grader: "I like solving hard problems best of all. When you solve problems like that, you get a special kind of stubbornness. It gives me great pleasure to sit over a problem and destroy one solution after another and then untangle the thread of the problem."[3] Finally, in vindication of the value, to a mathematics course, of solving problems by arithmetic, one must note the chief point: operating with numerical data ties arithmetic directly to life and creates the only solid basis on which the ability to operate with abstract mathematical symbols can be formed.

Solution of problems in general form, using letters to designate specified and unknown quantities, should be introduced as a generalization, after a number of problems have been solved with specific numbers. In teaching practice, however, algebraic generalization of the very richest factual material, which the children have been accumulating since the first grade, is put off for a long time.

Arithmetic and algebra thus are rigidly divided. Upon careful analysis of the present curriculum, one gets the impression that pupils in the primary grades are protected against the general conclusions for which the previous work seems to have made them completely ready. Such an attitude could only be justified if it were proved that making suitable generalizations is in fact beyond the ability of children in the first through the fourth grades.

Data from recent psychological and didactic investigations, however,

[3]From a manuscript by Polyak.

specially devoted to studying the cognitive possibilities of primary-grade pupils, testify to the direct opposite. These investigations cogently demonstrated that under the current instructional system (particularly mathematics instruction), the children's cognition is not used to the fullest. The children's capabilities for generalization and abstraction, moreover are particularly underrated. It was shown that this underestimation leads to artificial suppression of the pupils' development in the process of instruction. (See the works of Zankov and his collaborators--for example, [74].)

In connection with this one should naturally consider an earlier introduction of the elements of algebraic symbolism. Data from numerous psychological investigations show that heavier and more extensive reliance on generalizations in arithmetic instruction is one of the best means of increasing teaching effectiveness. Therefore there is every reason to suppose that elements of algebra, if introduced through a generalization of arithmetical facts, will help to raise the level of mathematical knowledge, and will, in particular, help improve the assimilation of arithmetic in the primary grades.

One of the most important problems of the psychology and methods of mathematics teaching, not yet solved decisively, is the system according to which arithmetical and algebraic questions should be considered; to what extent, in what form, on the basis of what exercises, and in what connection with arithmetical material can elements of algebra be included in the introductory course for the primary grades? It is perfectly clear that all these questions can be successfully answered only on the basis of experimentation and verification in wide school experience.

In recent years, experimental attempts at introducing elements of algebra in the primary grades have been undertaken by a number of investigators. Each attempt deserves the greatest attention, especially considering the importance of the questions raised and the role that the results of these investigations may play in the subsequent fate of mathematics education in our schools.

But before we consider the preliminary results of this research (the fifth section is especially devoted to this question), it is

necessary to inquire what shortcomings in teaching problem solving are observed in current school practices and what methods can rationalize the methodology of teaching problem solving, regardless of the answer to the question already discussed--how to introduce elements of algebra into the elementary mathematics course.

2. Basic defects in teaching problem solving

Two fundamental striking and closely-linked defects in teaching problem solving must be noted. These are the unwarranted wordiness of reasoning and the fact that the teacher implements it, thus condemning the children to passive imitation of this reasoning.

When solving a problem, pupils usually take the trouble to correctly reproduce a definite course of reasoning, rather than seeking and bringing to light the mathematical relationship hidden in the problem's condition. These shortcomings of practical instruction are, to a significant extent, predetermined by the method of presentation of the problem in our methodological literature, which gives primary attention to the teacher's role in explaining and analyzing problems.

Methods manuals and articles emphasize the way in which the teacher should read the text of the problem, briefly write its condition on the blackboard and explain it with a sketch or diagram. We are told in great detail how the teacher should analyze the problem, and so on. But there are extremely few directions about the pupil's role in acquiring the ability to solve problems by himself.

All the same, the method of the so-called "analytic" breakdown (from the problem's question to its data) which has been widely disseminated is one in which the teacher conducts the analysis himself while the pupils are required only to answer the questions the teacher asks; the questions are strictly regulated and are asked in the same stereotyped form (see Chapter III of [5]).

In this analytic method, a long enumeration of "what we know and what we do not know yet" artifically complicates the solution of a problem which can easily be solved by studying the condition: factual analysis of both the data and the unknown.

To ascertain the value of the analytic method, a number of investigations were performed by Kalmykova [28], Bogolyubov [3], and others.

It was established that when this method is extensively used, students fail to acquire skills necessary for independent analysis of a problem; they learn certain stereotyped forms of reasoning which they can apply only to problems which the teacher has already explained how to solve, or to problems like them. Skatkin was profoundly correct when he asserted as early as 1947 that the analytic breakdown of a problem (without synthesis) cannot be used to find a solution; at best it can only explicate the result after the problem has been solved [62, 63].

The solution of problems by this method is excessively prolonged (if, of course, we are talking about a problem requiring several operations). When a pupil reasons during the solution of a partial problem, he overlooks the remaining steps, ceasing to be aware of the problem's condition as a whole. Although when carrying out an analysis the pupil proceeds from the unknown, he gradually forgets this unknown in the course of solving. It is not amazing that, as a rule, pupils analyze a problem successfully only when the teacher continually interferes in the process, prodding them to perform the next step of analysis.

But there is one widespread negative feature in the teaching of problem solving that is also characterized as "verbal over-indulgence." It most often occurs in so-called "synthetic" breakdown (from the data to the question), in which the pupils are required to formulate a question each time they solve a problem ("We find out such-and-such from the first operation," etc.). But again, through the misuse of "verbal reasoning," the extent of the pupils' independent practice in solving varied problems is unjustly restricted, and it is this work that is most important.

As we will show later, both the elements of analytic breakdown of problems, and the formulation of questions before carrying out operations, are fully admissible in teaching practice, but under particular conditions. They should be used only within strict limits, without standardizing the procedure of solving problems, as commonly happens.

But the point is not just that these widely practiced methods of working on problems leave little room for the pupils to practice solving problems by themselves. There is the view that, in general, only those problems which the teacher has explained earlier in class (or analogous ones) may be given to pupils to solve without assistance. Even

one of the better methods manuals on teaching elementary arithmetic (Pchelko's) makes such a recommendation [48].

Under these conditions, try giving pupils in the elementary grades a problem they have not solved with the teacher. The overwhelming majority will refuse, justifying their refusal by the fact that "We haven't solved this kind of problem yet." Schoolchildren tend to develop the habit of thinking that they have to find out from the teacher how every specific problem is solved.[4]

With respect to problem solving by pupils, beginning in the lowest grades, an aim opposite to that which often occurs now should be created: One must think over the problem; since methods of solving it are unknown, one must find them by oneself. Only by systematically giving pupils practice in searching for ways of solving problems by themselves can they be taught to think, and thus can they form the ability to solve different kinds of problems.

What does ability to solve problems represent? Of what elements is it composed? How is it formed?

3. Analysis of the ability to solve problems

The ability to solve problems, like any other ability, presupposes knowledge of techniques (or methods) of carrying out operations. These operations should be executed according to definite, general rules, which promote a rational approach to a problem.[5]

What are these rules? Are they stated in the methodological literature, and do teachers use them? These rules do exist, but they are not spelled out in our methodological literature devoted to elementary mathematics. At the same time, every person who knows how to solve arithmetic problems conforms to these rules. Let us state several rules that are,

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[4]We have described the basic shortcomings in teaching how to solve problems in the elementary grades. This does not mean that worthwhile methods of teaching problem solving are never found in the experience of individual teachers and methodologists. We will repeatedly refer to this valuable experience in subsequent exposition.

[5]This does not mean, however, that the pupil cannot solve a problem before he has become aware of methods of solving it. But we will consider as a full-fledged skill one that is grounded in a distinct awareness of these methods.

in our opinion, the most important:

1. Do not start calculating until you have carefully studied the condition of the problem as a whole:

a) in reading through the whole problem, pay particular attention to the question;

b) return to the problem's condition and select related data; it often helps to write out the condition briefly;

c) if problems of a familiar sort can easily be distinguished in a given complex problem, solve them; then the problem is less complex and will be easier to solve.

2. In solving a hard problem, use different methods:

a) first, try to imagine exactly what the problem is talking about. To do this it is helpful to modify the problem: replace large numbers with small ones, invent a similar problem from your own life or, the reverse, ask yourself what you know from the condition and what you need to find out in the problem, and try to reproduce its contents in mathematical language;

b) a drawing or diagram made in class can help greatly in solving a difficult problem. The drawing or diagram should express the correlations between the given and the unknown (the question). When making a drawing or diagram, check yourself continually and construct the drawing from what is said in the text of the problem. Watch for mistakes in your drawing. If you find a mistake, correct it at once;

c) once you start to solve, performing every operation with numbers, keep asking yourself what you have learned by this operation and whether it was necessary to perform this operation in terms of the question asked in the problem.

3. After completing the solution, return to the problem's question and check whether you can given an exhaustive answer to this question.[6]

Methods of introducing these rules need to be carefully tested in teaching experience. At present, such testing has begun in the V. I. Lenin School (Moscow, Lenin Hills). The teachers are asked to look for the most successful form of communicating the rules to the children. A fourth-grade teacher, A. P. Voronova, has linked communication of the rules to an analysis of mistakes in pupils' written work. Depending on the kind of mistake a child makes, the teacher writes in the child's notebook, "Read the problem's question carefully," "Write out the condition briefly," etc. [39, pp. 50-56].

Checking is essential in mastering any skill. In solving arithmetic problems it takes the following forms: The pupil constantly compares the solution with the problem's condition and question, recognizes a mistake if he has made one, rejects a wrong operation and replaces it with another, checks the result, and so on.

Every teacher knows that children's inability to solve a problem is characterized primarily by a complete absence of checking. The pupils in such cases ask some questions, do a series of arithmetical operations, but never ask themselves the most important question--whether everything they are doing is correct, whether it corresponds to the problem's requirements. And only the impossibility of doing a numerical operation can stop them, can hinder them from continuing their thoughtless and unchecked activity.

Developing the habit of checking one's own work, in schoolwork in

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[6] A few words are said regarding the significance of rules of this kind in [59]. This question is given much attention in a book by the Kaluzh methodologist Voronin [70]. He formulates a series of rules that determine successful problem solving in the secondary grades of school. Of the foreign literature, Polya's How to Solve a Problem [52], translated into Russian, should be mentioned. This book is of interest even for the Soviet teacher. Of primary importance in it is an attempt to list rules that facilitate mastery of effective problem-solving methods. But in terms of substance, many of the rules require revision and supplementation.

The rules we have formulated are based on the data from our investigations and teaching, in the Soviet school.

general, and arithmetic in particular, should be a very important task of elementary school education. In the middle and upper grades this problem is given much attention (See [19]), but success can be attained only if the children have been learning in primary school to check themselves in the simplest form of school and work activity.

Practical experience is needed to form any skill, but this experience can vary in character, depending on the complexity of the skill. Thus, to develop simple, specific skills (for example, arranging numbers correctly in written calculations), uniform exercises, which teach pupils the same system of writing numbers, are necessary. On the other hand, to develop more complex skills, the kind needed to solve problems, varied experience is necessary, in which the pupil encounters various conditions requiring him to modify the specific methods of solution being applied, to choose the most rational of them, to vary to a certain degree the general rules which determine a successful solution.

Though general, the problem-solving ability, like all other abilities, lends itself to development, but we require a special system of exercises which make the children want to think creatively and which interest them in solving example problems by themselves and therefore in searching for the most rational ways of solving them. Independent composition (and solution) by pupils of problems on numerical material from their environment, problems related to their practical activity, should occupy a large place in this system of exercises. Only if unassisted problem solving is organized, if difficulty in solving grows gradually, so that surmounting it is within the pupil's powers, will his interest in solving creatively and independently be sustained and eventually become a need.

The general atmosphere of the work of the class as a group helps greatly in this respect. The atmosphere should be one in which when the teacher systematically encourages the pupils to use various methods of solving, and prods them to look for the most rational methods. The pupils, experimenting on their own, should become persuaded visually of the advantage of some method which they have found, and then modify solving techniques of their own accord, striving to find and use the most rational.

As instruction proceeds, the system of skills which the pupils have to master is enriched sequentially. New skills are built onto those acquired previously. At various stages of instruction, according to the complexity of the problems being solved, a mental operation (or a number of them) needing special mastery by the children in their independent practical schoolwork comes to the fore. Problems to be solved should help to fulfill this aim. For example, when the first story problems are being solved in first grade (in one operation), the process of choosing an arithmetical operation becomes important. Therefore, special exercises should be created so that the pupils can learn this process practically.

There is another general question concerning the system of problem selection--the correlation between prepared problems and the ones the pupils compose by themselves. What place should the latter category of problems have in the total system? There is no scientific answer to this question, and in practice it is resolved in various ways.[7]

In the textbooks for the various grades, assignments for making up problems are introduced in the most diverse sections and in various relations to prepared problems. There is, however, a definite tendency in the textbooks to introduce problem composing after the children have solved an analogous prepared problem. The opposite approach has also found expression in the methodological literature. The Polish methodologist Elen'ska puts it into words quite pointedly. In speaking of how much easier it is for pupils to solve problems they themselves have constructed, the authors resorts to a figure of speech: "The more we know about how the knot was tied and the noose was tightened, the easier it is to untie the knot [16: 151]."

But it is impossible to agree with that attitude. In the first place, the person who tied the knot is not always aware of how he tied it, and, in the second place, in real life a person, as a rule, comes across knots to untie without knowing how they were tied. Of course it is easier for a pupil to solve a problem he himself composed, but solving is not the

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[7] Another question--how to select various kinds of problems to compose--is better elaborated. It is answered in a paper by the teacher and methodologist Solov'ev [65]. A few amendments to his proposed system have been introduced on the basis of psychological research (see [37], Chap. VI, and [31]).

basic objective here. And therefore one can by no means conclude that solution of composed problems should always precede solution of prepared problems.

From observation and psychological research we know that in the problem he composes, the pupil most often reflects a mathematical structure which he has mastered very well. But teaching should lead the pupils forward, create new difficulties and have them encounter more complex problems, and once they have unraveled them, they themselves can then "construct" at a new, higher level.

Thus, in a number of cases, problem construction should immediately follow solution of a prepared problem. But it is especially necessary to choose such a form when the teacher composes a problem before the children's eyes, enlisting their services. This is an intermediate form between a prepared problem and a problem made by the pupils themselves. It may precede solution of a prepared problem. Popova, in particular, recommends this form [56, 57].

The effectiveness of this form when work is begun on a problem, and also the feasibility of employing it at later stages, requires additional testing in teaching practice. (Elen'ska's book gives much attention to this form of constructing problems, but the principles of its use are not absolutely clear [16].)

4. Fundamental questions concerning the methodology of teaching problem solving

Oue goal is to describe briefly a methodological system of exercises which facilitate development of skill in solving arithmetic problems. In connection with this, our attention will focus mainly on determining the most expedient type of exercise, conditioned by the psychological nature of the skill being formed and by the nature of the mental processes which the pupils are to master. But the number of exercises of a certain kind, and their specific content, should be determined by the teacher himself, taking account of his pupils' level of preparation.

The specific exercises we shall cite should be considered only as illustrations and could be replaced with others. A teacher's initiative

should be unlimited in searching for the most effective teaching methods. We are considering the system of exercises suggested here in terms of various categories of problems, from the easiest to the hardest.

Mastery of problem solving consists of a number of skills which constitute a complex system. When a pupil advances to new, more difficult forms of problems, new, more complicated demands are made on his mental processes. And at the same time, mastery of new operations depend upon skills formed by solving simpler problems.

We shall be especially interested in the transition stages, when a skill begins to be formed. We shall especially consider solution of the first arithmetic problems (in one operation) in first grade, when children learn how to choose an arithmetical operation, and then the transition to solving original problems in the same grade, when the children have to choose not only the operations, but also suitable pairs of numerical data.

Later we shall consider the basic directions in which the complication of compound arithmetic problems proceeds, and accordingly, what new methods should be used to enrich skill solving problems.

It is first necessary to recount the experience children have with arithmetic instruction when they begin to solve arithmetic problems. Children have some practice in solving examples—by now they have added and subtracted. But since they did so with objects, and since these operations were imbued with solid, vital meaning, solving an example and solving a problem at this stage of instruction actually were fused into one form of activity. Now another educational goal arises—to show the specific character of a problem as distinguished from an example, to separate them in the child's mind.

The children do not grasp this distinction easily, as proved by the numerous mistakes in solving and constructing problems in the first stages of instruction in which the children are actually likening a problem to an example. Thus, in solving a problem, they often give a numerical answer without even having read the question formulated in the problem (indeed, in solving an example, the important thing was to get a numerical answer, and no questions were asked in it).

Accordingly, when the children compose problems, they include the

numerical result in the condition: "At an airport there were 10 airplanes, 4 flew off, and 6 airplanes were left," "Mama bought 10 apples; I ate one, and 9 apples were left," and so on.

It is perfectly clear that the teacher's basic task at this stage is to make the pupils more aware of the question when they solve a problem, to show visually its real meaning in their own experience, to separate the question from the data in their minds.

To this end, how should the first arithmetic problems be introduced? In answer to this question, methodology takes into adequate consideration one of the laws governing intellectual development, according to which mastery of abstract knowledge is possible through reliance on concrete or even visually effective experience. The integer and fraction concepts are formed at the initial stages of instruction through the children's perceptions and operations with objects. Mastery of an arithmetic problem takes a similar course. The first problems are dramatization problems, as Pchelko calls them in his methods handbook [48].

However, with a dramatization problem, in which all the actions are performed realistically, one must be careful from the very start that the children do not perceive the unknown and thereby separate it from what is given in the condition.

In an experiment by a Leningrad schoolteacher, K. D. Zaitseva, which is described in the Popova book [57], this prerequisite was observed when the first problem was introduced. In the problem, which the teacher composed in the presence of the children, the given was demonstrated to the children visually (three fish and one fish) while the unknown--the total number of fish--was hidden, since all the fish were dropped into a pail. Here Zaitseva clearly divides the problem into two parts, saying: "What you saw is what we know. That was the beginning of the problem." In regard to the unknown she asks, "And what do you not see?" Getting the proper answer from the children, the teacher continues, "We don't see how many fish in all are in the pail--that is what we don't know. That is the problem's question" [57:20].

It is not worth the teacher's while, however, to linger over dramatization problems. It is necessary to go on to solving problems with a text in which a description is given of everyday actions with objects in

the absence of both the objects themselves and real actions with them.

In this case, what is known in the problem is no longer what the pupils see directly but what they have learned in the problem. Accordingly, the quantity which they have not yet found—which the problem's questions asks—functions as the unknown. Again, the basic goal here is to attain in their minds a clear cut separation of the data from the unknown.

But to do this it is necessary to develop in the children the ability to imagine clearly what is being described in the text of the problem, to "reproduce mentally the process described in the problem," as Topor puts it, generalizing her wide experience in methods work in one of the districts of Moscow [66:30].

Many teachers give children the special assignment of imagining what the problem is telling, and to make this process easier, they tell them to close their eyes. Some teachers conduct a whole system of exercises aimed at teaching the children how to imagine clearly what is described. For example, a teacher at School No. 172 in Moscow, V. D. Petrovna, whose experience is analyzed and generalized by Kalmykova [26], conducts special "visual dictations" in the first period of instruction in first grade. The children are shown various objects, and then they have to draw them from memory, trying to communicate more exactly their arrangement, color, etc.

Along with the ability to visualize the operations and relationships described in a problem, the pupils need to be taught the reverse process—abstraction. Indeed, in solving a problem, they are required to know how to translate a large number of varied everyday actions into the language of arithmetic, reducing all their diversity to the four arithmetical operations.

To carry out this task successfully, special exercises correlating the operations of life with those of arithmetic are necessary. A good exercise of this kind is for the children to construct problems by analogy or in which they must use another verb. For example, if a problem about escaped rabbits has been solved, then the children can be told to make a similar problem about fish. If a problem with the verb "bought" is solved, the teacher can then ask them to make a similar problem but

without the word "bought." In this respect, assignments which require one to compose problems of variable content according to a given numerical formula are also useful ("Construct a problem in which 5 is taken from 10."). This kind of assignment is often encountered in the first-grade arithmetic book. When the teacher uses these assignments, he must demand the most diverse versions of everyday activities conforming to the given arithmetical operation. If, for example, the teacher devises a problem that includes the verb "ate" for the numerical formula "10 - 5," in addressing the class, he asks the pupils to construct many other problems with the same numbers, trying to make them use different verbs: "took away, lost, spent, went away," and the like.

When the pupils begin to read the text of the problem by themselves, a new goal arises--to teach the children, as Topor [66] points out, "to take pains to grasp the meaning of the text of the problem," and for this they have to have special exercises. We mean teaching children correctly to single out the question through a suitable voice intonation, not to omit certain important words of the text that influence the choice of an arithmetical operation, and so on. In this connection, the better teachers do not limit themselves to systematically giving the children practice in reading the text of a problem, but show them how important a correct reading of the text can be in solving the problem successfully. For example, V. D. Petrova often says to the children as instruction proceeds, "Now Valya read the problem badly and can't explain it," or "Katya got the answer wrong because she skipped the very important word 'more' when she was reading at home." Not limiting herself to these hints, Petrova teaches the children to watch how their classmates read the text and to make appropriate comments. Thus her pupils make remarks like, "She didn't read the word 'more' very well" "She said the numbers badly," etc.

Teaching the children how to analyze a problem's condition is of the most essential significance even in first grade. Ability to analyze the condition in solving one-operation problems presupposes discernment not of any one element, but of a whole complex (although still a simple one), that is, both data and question. Only through awareness of both components of the problem can an operation be chosen.

This is the new characteristic which distinguishes a problem from an example. To solve an example, it was enough for the pupils to pick out one element: they extracted one word from the entire condition and, guided by it alone, chose the operation (for example, "'less'—that means you have to subtract").

Above we were saying that to avoid such mistakes it is necessary to vary the wording in the conditions of problems expressed in direct form. But an even more powerful means of counteracting the occurrence of such mistakes is the solution of "indirect" problems, since in these one must choose operations the opposite of those directly indicated by the individual words of the text. According to the existing curriculum, problems expressed in indirect form are given to pupils in the second year of school. In the textbook, they are collected into separate groups containing a large number of problems of one kind. But it is better to follow another principle in analyzing problems, by giving the pupils problems in direct and indirect form, alternately. This will teach them to analyze the condition of a problem more thoroughly and will help to overcome their tendency to choose an operation on the basis of individual elements of the condition.

Analysis of a problem's condition should be the subject of special exercises, and this is possible only when the center of gravity is transferred from calculation to analysis of the condition. Today's textbooks do not provide material for such exercises. It must be selected by the teacher, or else the children will inevitably consider the point of the problem to be calculation, not analysis of its condition. Modern teaching practices often make children think this way. Pupils (not only first graders, but older ones too), striving to get to the calculation faster, hastily and superficially analyze the problem's condition (or fail to analyze it at all). They do not know how to refrain from calculative operations where the condition calls for it.

There are great and still totally untapped possibilities for promoting development of the skill of analyzing a problem's condition. First and foremost, it is necessary to give children problems not only for solving with numbers but also for another purpose--analyzing the conditions and choosing an arithmetical operation. There may be different versions of exercises here--giving the pupils a series of problems

and requiring them to indicate the problems which can be solved by a particular operation,[8] or presenting a pupil with a certain problem, requiring him only to indicate the operation needed to solve it.

This method is recommended by Popova, but to a different end--to check how the children use operation signs in problem solving. In this connection, the author requires them to write the answer along with the operation sign. To teach how to choose an operation by analyzing the condition, it is enough to have the children indicate the operation sign without writing the answer.

One should stimulate the children to analyze a problem's condition thoroughly in other ways, too, by having them find a situation in which it is absolutely necessary to pay attention to the question.

Solving problems with the same data but with different questions has great value in this respect, since it is then the question which dictates the choice of various operations:

1. "Vanya had 2 apples, and he was given 3 more. Then how many apples did Vanya have? (Solved by the operation of addition.)

2. "Vanya had 2 apples, and he was given 3 more. Then how many more apples was Vanya given the second time than the first?" (Solved by subtraction.)

The second problem belongs to problems in difference comparison, and this kind, according to the 1960 curriculum, belongs in second grade. However, teachers have the opportunity to go beyond the curriculum somewhat if the level of preparation of the pupils in the class permits.

But use of this kind of problem can be avoided without exceeding the limits of the curriculum, and with this in view one can resort to joke problems such as this:

Borya has 3 apples and Vera has 5 apples. How many apples does their grandmother have?

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[8] This method is employed by M. I. Moro, as well as by L. E. Zainkina and T. V. Titova, teachers at School No. 315.

I. K. Novikov employed this kind of problem successfully in School No. 110. After the children incorrectly performed addition to solve the joke problem, Novikov explained the mistake, focusing their attention on the problem's question. Children react very emotionally to a mistake made under these conditions, and subsequently they pay more attention to the problem's question.

In papers on psychology proof has been offered that the material learned best is the material that was the subject of the person's activity (child's or adult's); that is, material on which he worked actively. These facts demonstrate the importance of exercises requiring children to invent a question to fit a problem's condition. Active work in composing a question promotes awareness of its role in the problem. The first-grade text does have this kind of assignment (for example, see p. 42, exercise No. 90), but only a few of them.

Teachers could give pupils problems without a question earlier than is usually done, in certain cases limiting such assignments to a statement of a question, and not requiring a solution. Independent composition of problems without any limiting instructions ("Think up a problem") is used on a rather wide scale in teaching practice, but this kind of work can be made more valuable if along with it the pupils are told to "think up" or "construct" an example and a problem, directly alternating both these assignments. This will help children to distinguish clearly between an example and a problem and to learn to construct them according to the assignment.

Once the teacher is convinced that his pupils have mastered the skill of analyzing the condition of a simple problem and of choosing an operation on this basis, he may go on to compound problems. To check, it is necessary to use special assignments from among those cited above. In this respect, the most appropriate are the ones recommended by Popova [57]. Pupils are given a series of problems and must indicate what operation is needed to solve each problem. Having the pupils carry out this assignment in written form and having them put down the operation sign makes it possible to ascertain, in a brief period, to what degree the entire class has mastered the skill of choosing an operation.

A pupil's mastery of operation selection at the preceding stage now makes it possible to shift the focus to choosing the numerical data, and

a new system of selecting exercises should comply with this. First it is necessary to recall the law of educational psychology which we have used in considering a methodology for solving the first arithmetic problems. In going on to the development of a new skill, it is necessary, from the very first, to let the pupil sense what constitutes the novelty of the problem before him, to confront him with a new difficulty, having sharply detached the new conditions from those to which he has been accustomed.

It is necessary to forsee what mistakes children can make while reasoning from their previous experience in solving single-operation problems. In solving simple problems, the children did not ponder over what numerical data had to be correlated, since there were only two data in the problem's condition. To answer the problem's question it was necessary to correlate the data in the condition. It is perfectly natural that children should approach compound problems with the same idea. Thus, the teacher's aim is primarily to create conditions under which it is clear to the children that they need an additional datum not in the text of the problem, and therefore to retard or brake, so to speak, their natural impulse to solve the problem immediately in one operation.

A textbook in which a series of simple problems with a missing datum is given before the section introducing compound problems helps somewhat to prepare them for this. By doing these exercises, they are schooled in the idea that there may not be enough data to answer the problem's question.

How should the first compound problems be constructed? At present, this question is being investigated by many methodologists, who are taking stock of psychological data [37:249-299; 48, 56]. The analytic transition has an advantage over the synthetic. The first consists in expanding or breaking down a compound problem into two simple ones, while the less rational, synthetic way is characterized by joining simple problems into one--into a compound problem.

The choice of the analytic way is dictated by the necessity of showing pupils more vividly the specific character of a compound problem compared to a simple one. But it is necessary to choose a problem with a structure which facilitates comprehension of the most important features of solving a two-operation problem. That is, problems with the

impossibility of immediately answering the question, and the need for an intermediate datum. This requirement is satisfied by a problem which includes two heterogeneous operations—addition and subtraction. This is such a problem:

> There are 6 pencils in one box and 2 fewer pencils in another. How many pencils in all are there in the two boxes?

Here the contradiction between what can be learned by the operation of subtraction (2 more) and what is asked in the problem is brought out clearly, since the question ("How many pencils in all are there in the two boxes?") is oriented toward the opposite operation--addition.

How easy it is to calculate a missing datum in a problem of this kind is proved in various ways. It has been found that in solving these problems, children err less frequently by blending two operations into one (when, having completed the first operation, they feel they have already answered the problem's question). Furthermore, a specially organized educational experiment (M. I. Moro together with A. M. Logacheva and L. E. Zaikina, teachers at School No. 315) showed that by introducing a compound problem first, the distinction between a two-operation problem and a simple one is established.

In this case (in which an uncomplicated two-operation problem is given), let us fully apply the method of analytically taking a problem apart (from the question to the data). The teacher asks the pupils if they can find how many pencils are in the two boxes immediately. After a negative answer, he asks new questions: "Why?" and "What do you have to find first?" At this point it is very useful to obtain a detailed formulation of the intermediate question from the pupils. (For example, in solving the problem cited above, the pupils should state: "First you have to find out how many pencils there are in the second box.")

Along with solving compound problems of various structures (at first including heterogeneous operations, then homogeneous) it is necessary to practice exercises specially intended to develop the children's skill in telling a compound problem from a simple one. To this end, they should be given pairs of related problems, one simple, the other compound. So that the specific character of the new problems will be the focus of the children's attention, they should sometimes be given to the pupils not to be solved, but only so that the children

can establish whether the problem can be solved immediately with one operation, or whether it cannot be solved immediately and needs two operations for solution.

Gradually, after accumulating some experience in solving compound problems, the children need to be brought to the formulation of one of the rules we indicated above. In proceeding to solve a problem, it is first necessary to ascertain what has to be found and whether there are enough data in the condition to find it, or whether a new datum is necessary.

In teaching how to solve compound problems, the teacher must give attention to a certain feature of the structure of two-operation problems. Some problems contain two numerical data, one of which is used more than once. (Thus in the problem cited above, the datum "6 pencils" is used two times: to answer the intermediate question, "How many pencils were there in the second box?" and to answer the problem's question, "How many pencils in all were there in the two boxes?") Another category of problems (in two operations) contains three numerical data, none of which is used more than once. For example:

> During the summer, the children raised 13 hens and 7 ducks. They gave up 16 fowl to the kolkhoz. How many fowl did the children have left?

Problems in the second category are of interest to us because they require the pupils to choose properly two data out of three. It is the presence of the third datum that creates the possibility that the pupil will make an incorrect correlation, a "false synthesis."

This characteristic of the problems must be further utilized in a different form by complicating the possibilities for choosing, thereby helping the children to form the ability to restrain or "inhibit" extra syntheses. This has paramount significance in mastering the skill of solving sufficiently complex problems.

In this respect, it is very important to vary the arrangement of the numerical data in the condition, and along with the so-called ordered problems (the course of their solution coincides with the order in which the numerical data are arranged) to give the children problems to solve which are not ordered (in which the order of data and solution do not coincide).

An analysis of arithmetic books for the elementary grades shows that ordered problems predominate. However, the teacher has every opportunity to increase the number of non-ordered problems by using the problems in the textbooks but suitably changing the order of the data in them. For example, in the second-grade book, Problem No. 578 gives this ordered problem:

> The young Pioneers sent 50 kilograms of fruit to their friends. They put 20 kilograms of apples into a big box and 5 kilograms of grapes into each of several small boxes. How many boxes of grapes were there?

This can easily be made a non-ordered problem by changing the wording:

> The Young Pioneers sent their friends 20 kilograms of apples and several boxes of grapes, 5 kilograms in each. They sent 50 kilograms of fruit in all. How many boxes of grapes were there?

Experience shows that when a problem is worded in such a way, some pupils tend to correlate the numerical data next to each other. Thus, in solving this problem they may divide 20 kilograms by "5 kilograms in each." Mistakes of this kind bear witness to a low level of problem analysis. To prevent them, it is important to alternate ordered with non-ordered problems systematically.

It is also necessary to watch so that particular data are not presented in an invariable combination. If this happens, children tend to carry out a habitual synthesis without performing exhaustive analysis. They slacken their checking and may easily fall into error.

For example, the second-grade textbook (pp. 23 - 24) gives a series of two operation problems in increasing and decreasing a number by several units. In all of these problems, three quantities are considered. What does the first one equal? Of the second, how much larger (or smaller) is it than the first; and of the third, how much larger (or smaller) is it than the second? The third quantity (or both the second and the third) has to be found. In all cases comparison is conducted in just this order--the second with the first, and the third with the second. Under these conditions, the level of analysis declines, and as a result the children make mistakes in solving problems analogous to the ones they have just solved, but in which the third quantity is compared, for example, not

with the second, but with the first. One must therefore see that the problems vary in this respect.

To teach the pupil how to analyze a problem's condition, it is also important to give numerical data written not only in figures, but also in words. This forces the pupil to think about the meaning of the text every time and not be directed purely by externals, correlating data written as figures. This purpose is served by including in the condition a figure datum having a direct bearing on the problem's story and not used as a numerical datum in solving. There are problems of this kind in the textbooks, although in insignificant quantities. For example, problem No. 255 in the second-grade textbook:

> In 3 days the children glued 40 books for the library: the first day--15 books, the second day--14. How many books did the children glue on the third day?"

Problems of this kind perform the same function as problems "with extra data," and at the same time they are not artificial--this is their advantage. One can deliberately make still more "provoking" assignments. Write in number form a story datum not used in the numerical solution (as "3 days" is written in the problem cited above), and write in word form one of the numerical data necessary for solving the problem. If the pupils make a mistake even here, realization of the mistake they made will be very useful, helping to raise the level (acuteness) of analysis in connection with problem solving.

At subsequent stages of instruction, the possibilities are expanded of using a different kind of interesting problem which requires the children not so much to solve as to analyze the conditions. The shortcoming of problems of this kind, which are given in the textbook (just as in the fourth-grade text, a special section of "Interesting Problems and Exercises" is marked at the end), is that most of them are not simple enough for the children to solve without assistance, and therefore they lose their educational value.

At the same time, there are a number of problems which are both within the abilities of the children and very useful, since they prod them to analyze the conditions more thoroughly. We will cite only one specific example of this kind, the solution of which requires no calculations at all. Rather, it is enough to ponder the conditions: "There were 60

kilograms of grapes in two boxes. Sixteen kilograms were transferred from one box to the other. How many kilograms of grapes were there in both boxes?"

The children who are thinking, after carefully familiarizing themselves with the condition, answer without any calculating. "It's still 60 kilograms"; "It doesn't change," they say. By contrast, the other children perform calculations (60 - 16 = 44, 44 + 16 = 60), without even noticing that they get the same 60 kilograms they started with. Problems of this kind train pupils to refrain from hasty action until exhaustive analysis has been completed, and this has great significance in forming creative thinking.

Problems in three or more operations make greater demands on the pupils' analysis of conditions. Therefore the authors of the second-grade textbook are correct when they introduce a special section in which the solution to two and three operation problems is given, and the children are told to compare the solutions of pairs of analogous problems and answer why one of them is solved in two operations and the other in three.

After a certain amount of practice by the pupils in solving compound problems, however, the number of operations ceases to be a factor determining the degree of difficulty. The character of the relationship between data, the ease (or difficulty) with which the relationship can be exposed, (its more "hidden," or conversely, its more "open" character,) begin to play a decisive role in this respect. For example, the character of the relationship is completely clear in a problem solvable in three operations which says that so many fewer cucumbers were taken from the second plot than from the first, and so many more cucumbers from the third than from the second (see problem No. 436 in the second-grade textbook). A two-operation problem solved by the method of ratios is much harder for the pupils. Let us cite one such problem:

> Twelve lemons were bought for a kindergarten. Fifty kopeks were paid for every three lemons. How much did all the lemons cost? (See problem No. 407 in the fourth-grade textbook).

This problem lacks any words which directly dictate a correlation of numerical data (as in the first problem, in which "so many fewer" or

"so many more" is indicated directly). Only through analysis of the specific features of the numerical data (the relation of 3 to 12) can one uncover the relationship contained in the problem's condition.

Problems of this kind, which can be solved by special techniques, are marked in the textbooks and put into groups, and a note, apparently for the teacher, points out what type of problem the given group belongs to. The works of Menchinskaya [37] and Kalmykova [27] map out a system of work that ensures the children's formation of the concept of problem type. Let us indicate only the basic stages of this work.

At first, immediately after analyzing the method of solving this type of problem under the teacher's direction, the children are given various problems of this type differing from the one discussed only in details--the story, the numerical data. But the essential elements of the condition which determine application of a type-method of solving are always given the same form in these problems.

At the next stage, the teacher sets a goal for the children--to learn to distinguish problem-type when there are substantial differences in wording which concern an essential part of the conditions (for example, the customary wording of the form "the first number is so many times as large as the second" is replaced by "the quotient in dividing the first number by the second equals....").

Later, problems of this type are included in the structure of more complex ones, and finally, the work concludes by comparing problems of the given type to problems of other kinds having something in common with problems of the given type in the wording of the conditions or in the method of solving.

Thus, after the children learn to solve problems on finding two numbers from their sum and ratio, they are told to compare problems in which the difference and ratio are given. Although these problems are not stipulated in the curriculum, in this case one may utilize the directions in the explanatory note to the curriculum, which allows the teacher to expand the study of a certain topic if the proper conditions are present.

It is also quite useful to contrast a type-problem to non-type-problems which are similar in wording. For example, after solving a problem on finding two numbers from their sum and ratio, it is a good idea to

give a problem in increasing a number several times, whose wording includes the same expression "so many times as much." Indeed, introducing exercises of this kind into teaching practice aids the development of pupils' flexible skills in solving type-problems, by preventing the influence of a stock phrase in solving.

Alongside of problems which lead the pupil forward, we discover in the textbooks a number which in no way enrich the pupil. These are peculiar "intervals" and "pauses" filled with other forms of activity, during which problem solving acquires especially subordinate significance and the structure of the problems which the children are told to solve is purposely simplified, although they have already learned a more complex structure. This occurs, for example, in the first-grade textbook in the transition to studying the new operations, multiplication and division. Two operation problems, which the first graders have already solved, almost completely disappear in this section. They appear only in the section "Practice," and then do not appear until the second-grade textbook.

The authors of the textbook gave the following reasons. Only simple multiplication and division problems are introduced into the curriculum of the first year of school. These two new operations of arithmetic are studied in the second half of the year, and the selection of problems is entirely subordinate to the objective of studying these operations.

Such "intervals" are met again and again. In first grade the "Multiplication" and "Division" sections include only single operation problems, and the same thing occurs in a substantial part of the section "Numeration up to 100." It is natural that the problem structure is also greatly simplified in the study of multiplication and division tables; in the fourth-grade textbook, for a while the numeration of numbers of several figures, and operations with them, supplant arithmetic problems of a more complex structure, and the same thing occurs in the study of concrete numbers, and so on.

What does all this say? It testifies to the fact that the system of teaching problem solving is very often violated. And it is well known that every skill can be formed successfully only if systematic exercise is ensured.

How is one to get out of this situation? How should the teacher

proceed with the textbook material? The teacher should not slavishly follow the textbook in selecting problems. He must strive to ensure a definite system in solving problems, gradually increasing their difficulty and reinforcing acquired skills. Therefore, in first grade in particular, when working with children on single operation problems in multiplication and division, it is necessary constantly to alternate these problems with compound problems in addition and subtraction. Then the teacher does not have to worry that the ability to solve two operation problems, acquired not without labor in first grade, may be destroyed because of a prolonged lack of practice in solving them and that it will have to be re-formed in the second year of school.

Moreover, if the pupils' preparation permits, the teacher can go somewhat beyond the bounds of the first-grade curriculum, giving compound problems to solve that include not only addition and subtraction, but also multiplication or division (along with the simple problems in the text). The explanatory note to the curriculum grants a teacher this right.

The teacher can proceed in a similar way with type-problems, sometimes deviating from the textbook, and if expedient, going somewhat beyond the bounds of the curriculum. When students independently, though with the teacher's help, "discover" a type-method of solving (and this is what the methodology of explaining type-problems should be), they do not, as a rule, need to solve as large a number of analogous problems in a row as the textbook gives (25-30). It is considerably more useful to move some of these problems to other sections, alternating them with problems of simpler structure, related to studying some new question of arithmetic. Solving a type-problem some time after it has been introduced will be very useful, since it requires the pupil to analyze the problem's condition a second time and will not be simply the result of recalling a specific method repeatedly used earlier in solving analogous problems.

In teaching how to solve compound problems (which demand considerable mental effort from the pupils), the pupils' awareness and mastery of methods of solving them, and knowledge of the general rules by which they must act, acquire still greater significance (than when the solving of simple problems was being taught).

The methods with which the children were acquainted at the initial stages of instruction (concretization, abstraction, and the like) acquire richer substance; now the primary object should be the use not of individual methods but of the whole system of them. Children should be taught to choose methods correctly depending on the nature of the problem.

We have seen the content of one of the most important mental operations--the operation of choosing--gradually deepen. If in the beginning the pupil has been required only to choose the arithmetical operation, and then to choose numerical data to correlate with each other, now in addition, he must choose methods which ease the task of finding the path of solution.

The question may arise, is this latter task within the powers of primary-grade pupils? The psychological research that has been conducted allows this question to be answered in the affirmative.

According to Kalmykova's data [26], many pupils could be found in third and fourth grade who actively searched for methods of solving unfamiliar problems. They resorted to constructing a diagram illustrating the problem's condition, or they tried various type-methods they knew, or they varied one of the data to ascertain how it would affect the other data. They did this even though they had received no special instruction in a system of methods of solving problems independently. There is good reason to suppose, therefore, that with such instruction, children can learn to choose more effective methods according to the character of the problem.

It is important to teach children, for example, to resort to the method of making a condition concrete if the problem is stated abstractly, or, the reverse, to use the method of abstraction for a problem with a concrete subject. And finally, it is particularly important to teach them to try out different methods, replacing one with another in case of failure.

This latter objective can be realized only through giving pupils extensive practical work in solving problems independently. The number of problems children solve independently should be substantially increased at the expense of problems solved with the teacher's direct help and guidance. What should be the content of basic methods for solving relatively difficult problems independently?

The method of making a problem's condition more concrete should be applied in more active forms than at the initial stage of instruction. Then the teacher limits himself merely to advising the pupil to picture clearly what the problem said. The necessity of a vivid "image" of what is described in the problem retains its significance even now. But it should be taken into account that one must know how to use images. A vivid image of individual, complex details of a problem's condition may even play a negative role, leading the pupil away from discovering basic correlations. And, most importantly, this process is hardly subject to control.

Other forms of concretization, linked with active alteration of the text of the problem, are much more convenient. It is very useful to advise the pupils to alter a problem given in abstract terms, if necessary, into a problem with a specific story. For example, if a problem is given about a sum and ratio in abstract form:

> The sum of two numbers is 12, and one number is 3 times as large as the other. What are the numbers?

the children themselves can make it into a problem with a specific story:

> Two boys had 12 pieces of candy. One had 3 times as many pieces as the other. How many pieces of candy did each boy have?

In certain cases, temporarily replacing large numbers in the problem's condition with small ones is useful. Such substitutions are a particular form of concretization, since they help to bring the problem closer to the pupil's experience. It is necessary to give the pupils special exercises, telling them to alter the conditions of problems, and then to prod them to use this method as they search for ways of solving an unfamiliar, abstract problem by themselves.

Interpretation of a problem's condition is aided by the opposite method--abstraction--in which the condition of a problem with a concrete subject are formulated in the language of abstract mathematical correlations of quantities. Not enough attention has been given to this aspect of reinterpreting a problem until now. Still, this kind of work is very important, since it trains the pupils to "translate" into mathematical terms various correlations clothed in the concrete form of actions

of life without assistance.

Bogolyubov, a teacher, worked out an entire system of such exercises [4]. Several problems were given, varied in subject matter and identical in mathematical structure. The pupils are to replace words denoting various everyday actions ("bought so many," "spent so much," and the like) with one and the same word, which has the nature of a mathematical term, "cost." From these abstract terms the pupils easily pass to the arithmetical operation, since the relationship between quantities becomes completely clear to them.

Bogolyubov [3], in generalizing his many years of teaching experience, gives a "scale" which reflects the different degrees of approximation to abstract mathematical language. He shows this by the wording of one and the same final question of a problem:

1) the most concrete wording: "How many mushrooms did the boy and girl pick together?";

2) more abstract: "What number of mushrooms did the boy and girl pick?";

3) still more abstract: "Find the total number of mushrooms.";

4) the most abstract, completely free of all aspects of "plot": The problem is to find the sum of the numbers.

In this last case, the content of the problem is virtually reduced to naming the arithmetical operation with which the problem should be solved.

Doing such exercises gradually makes it possible for the pupil to use abstraction as an independent method for solving problems. This is facilitated by the textbook, which introduces exercises in composing and solving problems, in which the data are given in the form of abstract concepts: "number," "cost," "value," and the like.

More effective and advantageous for wide-scale use in school practice is a method which unites concretization and abstraction. We refer to the use of graphs and diagrams which help show the correlations between the quantities mentioned in a problem. A graphic illustration, on the one hand, makes it possible for the pupil to picture these correlations in visual form (concretization), and conversely, it helps him to abstract

himself from the details of the story and the objects described in the text of the problem (abstraction). This is the basic value of the graphic method.

In practice in our schools, graphic illustrations and diagrams usually are implemented by the teacher himself. Very little has been done till now to make graphic, schematic representation of the condition the unknown, and the link between them, a method of independent problem solving by the pupils. A diagram does not show the objects discussed in the problem: It presents mathematical relations between quantities in abstract form. This is why it is hard for a child with no special preparation to grasp the connection between an abstract sketch and concrete conditions.

One psychological investigation showed that the use of drawings of the kind given, for example, in the second-grade textbook on pages 7, 16, 18, 22, and others, in the fourth grade textbook on pages 63, 79, and others, is good preparation for employing diagrams [6]. In these drawings (which Botsmanova called "object-analytical") the pupil sees the objects discussed in the problem. These drawings reflect the quantitative correlations between data and unknown. Such a drawing is closer to the text of the problem and is distinguished by greater concreteness than a diagram, and in this respect it is easier for the children to understand. However, to be limited to the use of such drawings would be utterly wrong. Just because of their concreteness or "attachment" to the story of a definite problem, they can do little to help the child understand the general method of solving problems of the same kind.

A diagram revealing links between quantities which are veiled in the text of the problem is more likely to lead the pupil to discover a way of solving the problem than the object analysis drawing. Therefore, it is very important to teach children to use diagrams in solving problems. Very useful in this respect is the method of contrasting an object analysis drawing and a diagram of one and the same problem. In this case, the object analysis drawing plays the role of an intermediate link. By contrasting the text of the problems, the drawing, and the diagram, the pupil becomes aware not only of the concrete correlations described in the given problem, but also of the connection between the quantities considered in it in a general scheme. For example, the link between cost, price, and quantity in general is shown, regardless of what objects are

being talked about.

Research shows that ability to construct a diagram develops in children by degrees. At first, they grasp and express in the diagram the most general features of the problem, and only afterward do their diagrams begin to reflect the structure of the problem. Making a diagram is not an end in itself. In constructing it, the pupil gradually perceives the relationship contained in the condition. He continually compares his drawing with what is described in the condition and makes corrections if necessary. Consistently perfecting the diagram means perfecting the analysis and raising the level of analytical work.

When pupils become convinced by experience that using a diagram aids successful problem solution, they begin to apply this method on their own initiative on a large scale, without waiting to be prompted by the teacher. Those teachers who systematically teach problem solving with the graphic method achieve good results. For example, A. V. Olevanova, a teacher at School No. 330 in Moscow (whose experiment was studied by Botsmanova), who uses the graphic method extensively in teaching problem solving in first and second grade, gave children problems of increased difficulty to solve by themselves, which went far beyond the limits of the curriculum, and the children coped with these problems successfully.

Certain methodologists [61, 67] give much attention to elaborating the graphic method. However, it has not yet found proper reflection in the arithmetic textbooks. Though we could cite a number of examples of object analysis drawings which are available in the books, we can hardly do this with diagrams. The only exception is specific diagrams for motion problems. At the same time, the textbooks have many illustrations for quite another purpose: to show the meaning of a word used in the text of the problem. These drawings often concern things sufficiently familiar to the children, so they may be discarded without detriment.

In speaking of the use of diagrams in teaching problem solving, one inexpedient form should be mentioned, albeit briefly. We mean the construction of diagrams in the course of a so-called "analytic" breakdown of a problem (from the final question to the data). Popova often recommends that such diagrams be used even when the problem itself is easy. At the same time, the construction of a diagram of the analytic breakdown is distinguished by certain specific difficulties [56: 105]. However,

the diagram here is an end in itself, a certain special exercise in logic, whose psychological and educational value is at least questionable.

Popova proposes another principle of diagram construction in her work on methods of teaching arithmetic in first grade [57]. The diagrams she proposes for first graders help to show the correlations between data and the unknown. But according to the directions in this book, the most active role in constructing diagrams still belongs to the teacher, not to the pupils.

All of the methods described above were directed at analysis of the conditions of a problem and exposing the relationship between the quantities discussed in it. The use of the analytic question "Why?" belongs to this group of methods. Utilization of this method is possible only for certain kinds of problems. Thus, in solving problems "in finding a number from two differences," the statement of the question, "Why was the cost higher in one case then in another?" for example, helps to "untie" the basic "knot" of the problem, since analysis, naturally, aims at exposing the difference in the number of objects which causes the difference in cost.

Whether or not unassisted use of this method is within the pupils' abilities was tested in an experiment by a teacher, A. E. Kozlova, in connection with teaching third graders [32]. But the pupils must be gradually led to unassisted use of the method. In the beginning, the teacher himself poses the question "why?" in dissecting a new problem, and later he tells the pupils to ask the question in analyzing a problem which is familiar to them; finally, he prompts them to use this method in analyzing a problem which is new to them. In this case, the pupils themselves ask a question aimed at exposing the basic relationship, and they themselves answer it and independently find ways of solving the new problem.

Analysis of the relationship between data can be approached a different way--by arbitrarily changing one of the data and then ascertaining how the change affects the other data. This also helps to show the relationship between basic quantities. Certain exercises, widely used in practice, can help pupils to master this method. We mean assignments which call for transforming the problem into another, when the question of a problem which has already been solved is changed and the pupils are

required to change the condition, or the reverse, to think up a new question for an altered condition. Exercises of this kind have been worked out by Polyak [53]; they also occupy a large place in Pchelko's methods manual [48].

If analysis of the problem's condition by the methods described above does not lead to the goal, there remains one method--a less perfect one, which, however, children often resort to on their own: performing a numerical solution as a "trial." These trials can be effective only if the pupils consider them as steps toward deeper analysis of the problem. In other words, if any pair of data are correlated and a numerical solution is produced, then right away the question must be asked--can one correlate these data? does this not contradict the other data? and would it be necessary to find this out, from the standpoint of the problem's question?

In using this method, there is always the danger of transforming the solving process into mindless manipulation of numbers, which some pupils who are not fond of hard problems are prone to do. Therefore, use of this method by pupils must be approached with great caution.

Carrying out a numerical solution before exhaustive analysis of the problem is justified only if the pupil knew how to expand the problem into a series of simpler ones during preliminary analysis, first isolating a familiar problem from one whose methods of solving had to be found. By progressively solving such familiar, simpler problems, he simplifies the complex problem given him, and thereby can direct all his thinking efforts toward solving the basic problem.[9]

Thus, teaching the pupils methods of solving problems independently by practice should be the basic objective of the methodology of teaching problem solving. But along with elaboration of the most effective methods for pupils to solve problems, one must consider another aspect--a reasonable simplification of techniques of working on problems. Some firmly established requirements could be considered optional, or even superfluous.

For instance, in solving problems, must we insist that the pupil

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[9]The technique of expanding a problem into simple problems is described by Shor [61].

write down the concrete name of every component in an operation? Is it not enough to write the concrete name in the result? As we know, much time is spent getting pupils to learn two different kinds of division--"into parts" and "by content" and having them master the two corresponding ways of writing concrete names in solving problems.

An observation shows, even if teachers contrast these two problem forms, a number of pupils will continue to make mistakes in writing the concrete names. Is the expenditure of time and effort on this kind of schoolwork justified? Apparently, it is often enough to require the pupil solving the problem simply to explain his result briefly, or even merely to correlate his answer with the question asked in the problem.

Similarly, legitimate doubt arises as to whether it is advisable always to require the pupil who is solving problems in two operations (or more) to analyze the process into isolated steps, when it would be more economical to write the necessary operations on one line. We refer primarily to the kind of problem which can be solved either by double subtraction or by addition and subtraction.

One often encounters problems like the following:

> Some schoolchildren had 10 rabbits. They gave the kindergarten 3 rabbits, and another rabbit ran away. How many rabbits did the schoolchildren have left?

The children solve the problem very quickly, successively taking 3 and 1 away from 10, while the teacher is trying to get them to formulate two separate questions and to perform addition first and then subtraction. One asks, should children be hindered from using the simpler way?

In connection with subsequent improvement of arithmetic teaching procedure, two objectives should be established: 1) to simplify teaching by removing from the arithmetic course everything retained in it for tradition's sake alone, and 2) to introduce more complex academic material. Attempts to reorganize the subject matter of the elementary mathematics course, undertaken in recent years by a number of researchers, are moving in just this direction. We go on to consider the preliminary results of these experiments.

5. New experimental research considering the possibilities of introducing elements of algebra into the elementary mathematics course

The first experiment in introducing elements of algebra was performed in the experimental class of El'konin and Davydov [14]. Unfortunately, in the published materials no special attention was given to teaching children problem solving. But from the outlined curriculum [14] it can be seen that the experimenters propose that children as early as second grade begin to construct and solve first-degree equations in two unknowns in connection with the solution of problems. There is still no data showing whether the experimenters succeeded in realizing this program in practical teaching, and if so, under what kind of system the corresponding work was conducted.

Substantially more material on the question under consideration is contained in an article by the Ukrainian psychologist, Skripchenko [64]. During the school year 1960-61, Skripchenko conducted some interesting work on developing in fourth-grade children generalized algebraic methods of solving problems. Convinced that this material was not beyond the powers of fourth graders he transferred his experiment to the third grade during 1961-62.

The preparation for solving problems by equations is described in the following manner:

> Before going on to setting up equations from the conditions of problems, the pupils distinguished and elaborated individual steps in this complex process. In the beginning, they formulated individual algebraic expressions from statements representing the conditions of problems [64: 87].

It would be more accurate to say that they learned to express one of the unknown quantities in terms of another when their difference or sum was known. For example, the children are told to formulate algebraic expressions according to this condition:

> On the first day, a worker made several machine parts, and on the second day he made 5 more than on the first day.

Designating by x the number of parts made by the worker on the first day, the pupils then formulate an expression $(x + 5)$ to designate the number of parts made on the second day.

Then the children went on to problems in several unknowns. The authors states:

> After the pupils had developed elementary skill in constructing individual algebraic expressions from the statements functioning as parts of the problem's condition we taught the children to join individual expressions into equations—judgments. This work was carried out on problems similar in kind, then on problems varying in content and structure [64: 89].

Later, by contrasting the arithmetic method of solving problems with using equations, a link was established between the skills developed in the children earlier and the new method of solving. The general conclusion the author makes from the experiment is that

> Methods of solving arithmetic problems by setting up equations in one unknown can be learned by pupils in the primary grades--not only fourth grade, but third grade also [64: 92-93].

From this summary, it can be seen that Skripchenko's experiment envisaged introducing the algebraic method of solving problems earlier than in sixth and seventh grade but in approximately the same way. The general direction in teaching the formulation and solution of equations was maintained: in the first year, solving problems by the arithmetic method; later, constructing simple algebraic expressions reflecting the problem's individual elements; and finally, constructing equations. The algebraic method of solving problems is introduced on the basis of the arithmetical one and in contrast to it.

The proposals formulated by Pchelko took approximately the same direction [47]. But he approached the solution of the question more cautiously. Pchelko wrote:

> Of the two basic methods of solving problems, the arithmetical one will occupy the leading position in the primary grades under all conditions, because the arithmetical structure of the majority of problems solved in primary school is such that it makes use of the algebraic method superfluous [47: 68].

As an example for which "the algebraic method simply does not make sense," he cites this problem:

Two trains left two cities at the same time, heading toward each other. One was going 40 kilometers an hour and the other, 60 kilometers an hour. The trains met in 4 hours. Determine the distance between the towns. [47: 68].

At the same time Pchelko considers it expedient to use equation construction in fourth grade to solve simple problems on finding an unknown minuend when given the subtrahend and the difference, finding two numbers from their sum and ratio, and certain others. He proposes that the very simplest equation, such as these should be solved on the basis of knowledge of the relationship between components of the arithmetical operations. Since these relationships are at present "studied fundamentally" only in fourth grade, Pchelko considers it possible to introduce suitable equations only in that year, in order to use the algebraic method of solving in fifth grade for complex, "intricate" problems.

Thus even here, as with Skripchenko, it is a matter not of changing the system of teaching problem solving as a whole, but only of introducing equations a little earlier than is done now, in order to make it easier for pupils to solve more complex problems and to save some of the time it takes to cover the curriculum.

Another approach to the question under discussion was outlined in an article by the authors of this book [40]. We proceeded from the fact that the algebraic method of solving shifts the focus of attention from the calculation process to the analysis of the relation between data and unknown, and that such a method requires comprehension of the mathematical structure of the problem as a whole (in the arithmetical method the pupil's attention is often taken up by individual, particular problems and the whole is overlooked).

The algebraic approach to problem solving thus supposes a higher level of generalization, and makes greater demands on the ability to perform analysis and synthesis, than the arithmetical approach. Therefore conscious mastery of the algebraic method of solving complex problems should be preceded by special (prolonged) preparatory work, the aim of which would be to gradually instill suitable skills in the children.

This is why it seemed necessary to us to conduct such an "algebraic preparatory course," beginning in first grade. In the teaching experiment organized during 1962-63 under our direction in the first grade of the V. I. Lenin School (R. F. Sen'kina, teacher), this preparatory work

was begun from the moment the children were introduced to solving very simple addition and subtraction problems [39]. The children were systematically taught to discern closely what is known from the problem's conditions and what is not known—what has to be found. In the notation for solving these problems, x was introduced immediately to designate the unknown number, and the children had to learn to express the unknown in terms of the known quantities with the aid of the signs of the arithmetical operations.

Let us show by examples how the pupils themselves explained the course of their solutions. Repeating the problem given by the teacher, the pupil immediately isolated the known from the unknown. For example:

> We know that Kolya had 5 stamps and that his papa gave him 1 more stamp. We have to find out how many stamps in all Kolya had then.

Further reasoning goes like this:

> I call the unknown x--that's how many stamps Kolya had in all. We know that Kolya had 5 stamps and his papa gave him one stamp. So he had more stamps. We need to add 1 more stamp to the 5 stamps. I write $x = 5 + 1$. Now I'll figure out what x is (calculates mentally and writes), $x = 6$. Kolya had 6 stamps in all.

It can be seen from this course of solution that here (as in the problem involving "oncoming traffic" cited in the Pchelko article), neither the introduction of x to denote the unknown, nor the construction of an equation, by themselves made the problem any easier to solve (at first it even perhaps complicated the pupils' work somewhat). However, as we said above, it is necessary to evaluate the expedience of such work according to its use in preparing students for mastering the algebraic method of solving complex problems. From this standpoint it is highly significant, since with such an approach, from the very beginning, problem solving is clearly divided in the children's minds into the following stages:

1) discerning and delimiting the data and unknown;

2) designating the unknown by x and writing the relation between unknown and data in the form of a definite mathematical expression;

3) finding the numerical value of the unknown.

These are all very important aspects of preparation for setting up equations.

Even in the first grade, the children were introduced to so-called "reverse" (or "indirect") problems in adding and subtracting (problems involving finding one of two addends, given the sum and the other addend; finding the minuend, given the subtrahend and difference, and others). Problems like these were part of the first-grade curriculum in the past but were transferred to the second grade in 1950 because of their extreme difficulty for seven-year-olds. The expediency of this decision raised doubts from the very beginning. At that time, in 1950, Menchinskaya observed that "difficulty in learning is no argument" and that "we should have striven to overcome the difficulty first graders had in learning [36: 73]," taking into consideration the psychological value of these problems for education in this first year of school. This consideration was subsequently reinforced by an analysis of the difficulties linked to studying problems expressed in indirect form, in second grade. The analysis showed that rejecting these problems in first grade regularly leads the children to adopt an incorrect approach toward problem solving in general [43].

There was every reason to suppose that solving reverse (indirect) problems in first grade would facilitate development of the children's ability to grasp the problem as a whole and to choose an operation consciously on the basis of adequate analysis of the conditions. Considering direct and reverse problems at the same time creates conditions which preclude the development of stereotyped solutions and which do not permit children to establish a mechanical tie between certain isolated words in the text of the problem and a particular arithmetical operation.[10]

There was another purpose in introducing problems of this type into the experiment, that of gradually preparing the pupils to use the method

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[10] These considerations were taken into account in an experiment performed under the direction of Zankov [73]. In connection with teaching problem solving in first grade, this experiment attaches special significance to reverse problems and comparison of them with direct ones.

of setting up equations. To this end, in solving reverse problems, x was used to designate the unknown. The solution of a problem on finding one of two addends from their sum and a second addend, for example, was written this way:

$$x + 3 = 7; \quad x = 7 - 3; \quad x = 4$$

The transition from $x + 3 = 7$ to $x = 7 - 3$ was accomplished by analysing the concrete life situation described in the problem. For example, this problem is solved:

> There were several pencils in a box. The teacher put in 3 more pencils. Then there were 7 pencils in the box altogether. How many pencils were there in the box at first?

The pupil explains that to answer this question, one needs to take all seven pencils and lay aside ("take away") the three pencils the teacher put in. Then the pencils which were in the box in the beginning will remain. By solving problems of this kind many times and carrying out the reverse assignment of composing problems from a given solution, the pupils accumulate factual material which prepares them for realizing in general form the link between components of arithmetical operations. Consciousness of these links is a prerequisite for solving first-degree equations in one unknown.[11]

Preparing first graders to make equations was not limited, however, to solving simple problems. Changes were introduced into the methodology of solving compound problems. Let us show this with an example.

> There were 8 pieces of candy in a bowl. Four pieces were eaten. Mama put 5 more pieces in the bowl. How many pieces of candy were there in the bowl then?

The pupils in the experimental class reasoned and wrote the solution to

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[11] The feasibility of using x in solving reverse problems in first grade, and the feasibility of teaching first graders a general method of solving these problems, was demonstrated in Kossov's work [30]. This is borne out by the experiment conducted in the schools of the Sverdlovsk region under the direction of E. M. Semenov, although both authors approach formation of such a generalized solving method somewhat differently than we do.

this problem in the following way:

> It is necessary to find out how many pieces of candy there were in the bowl then. I'll call that x. We know that there were 8 pieces in the bowl, but 4 pieces were eaten--so there were fewer—you have to take away 4. Mama put another 5 pieces in--you have to add 5. Then $x = 8 - 4 + 5$. I'll calculate what x equals (calculates and writes down $x = 9$). There were then 9 pieces of candy in the bowl.

By such a method of analysis, the pupil mentally grasps the entire solution of the problem as a whole before he proceeds to calculate.

Other forms of compound problems were approached in a similar way. In order to form the children's capacity to analyze and synthesize, generalize, abstract and "concretize," techniques of analyzing a problem's condition, such as schematic and object illustration of the condition, we constructed, or had them construct, diagrams reflecting the link between data and unknown, and others. All of these techniques were used not only by the teacher, but by the children also.

To prepare the children for algebraic generalizations in the later grades, we gave them, after they had solved a large number of problems with numerical data, analogous problem questions without numbers, in which they had merely to indicate what operation of arithmetic was needed to answer the question. For example, the teacher would say, "If you know how many books are on one shelf and how many are on another, then what operation do you use to find out how many books in all there are on the shelves?"

The first experiment conducted in this direction, in the V. I. Lenin School in 1962-63 gave positive results. The above mentioned innovations had a beneficial influence on the way the children learned the entire curricular material, and not only permitted the curricular limits to be exceeded, but also significantly raised the theoretical level of pupils' assimilation of mathematics study material in first grade.

Our experimental program calls for, in particular, implementation of the algebraic preparation, in connection with the instruction in problem solving which was considered above for first grade.

In the next grades, this work receives further development. Thus

in the second grade, in generalizing the vast factual material accumulated during their first year, the children learn, in general form, the relationship between the components of the arithmetical operations. On this basis, they solve problems which are essentially elementary first-degree equations in one unknown.

In third grade, they become familiar with using the method of equation construction in solving problems such as:

> A housewife bought 3 kilograms of potatoes at 10 kopeks per kilogram and 2 kilograms of cabbage. She paid 62 kopeks for the whole purchase. How much does 1 kilogram of cabbage cost?

The equation constructed from the problem's conditions can be solved by knowing the relationship between the components of the arithmetical operations. The pupils reason this way:

> The housewife paid 10×3 kopeks for the 3 kilograms of potatoes, and $\underline{y} \times 2$ kopeks for the 2 kilograms of cabbage, and the entire purchase cost 62 kopeks. Then $10 \times 3 + \underline{y} \times 2 = 62$. To find the unknown term, it is necessary to subtract the known term from the sum: $\underline{y} \times 2 = 62 - 30$, $\underline{y} \times 2 = 32$. To find the unknown factor, the product has to be divided by the known factor: $\underline{y} = 32 \div 2$, $\underline{y} = 16$. One kilogram of cabbage costs 16 kopeks.

Other problems which reduce to constructing and solving first-degree equations in one unknown can be solved in the same way.

Furthermore, beginning in second grade, along with solving problems constructed on specific numerical material and problem questions without numbers (discussed above), the children become familiar with solving problems in a general form, in which the assigned and unknown values are designated by letters. Problems like this are solved:

> The first day, $\underline{A}$ kilograms of potatoes were sold, and on the second day, $\underline{B}$ kilograms. How many kilograms of potatoes were sold in the two days? or, There are $\underline{A}$ students in two classes. There are $\underline{B}$ students in one class. How many students are there in the other class?
>
> (Solution: $\underline{x} = A - B$.)

Problems of this kind, formulated and solved in general form, are made specific by the children, who give $\underline{A}$ and $\underline{B}$ distinct numerical values.

In the experimental classes, exercises in independent construction

of problems, from a prepared solution and from a schematic or brief notation of the condition, are systematically conducted. After problems with letter data have been introduced, problems are constructed according to formulas as $\underline{x} = A + B$, $A + \underline{x} = B$, $x \cdot A = B$, and the like. These problems are constructed both in general form and with numerical data (depending on what the teacher assigns). The reverse assignments are also given. For example, a problem with numerical data is used to set a general form.

All these exercises, as observations show, help the children to become aware of the features of a problem. They begin to identify what innumerable distinct, concrete problems which are mathematically identical have in common, and the basic differences which determine the choice of different arithmetical operations in solving problems which appear quite similar at first glance, for example, solving a pair of problems involving finding the sum of two addends and finding one of the addends from the sum and the other one. They are good training for studying how the components of arithmetical operations are related.

The use of algebraic symbols thus promotes more thorough mastery of arithmetical material. The exercises described, moreover, are found to be very useful in developing children's capacity to generalize, abstract, and concretize. In this respect, the use of letter symbols has proved successful in considering the basic properties of the operations of arithmetic.

In second grade the children learned, with full comprehension, to write the commutative property of addition and multiplication in letters, and in third grade--the distributive property of multiplication over addition, various methods of adding (and subtracting), the associative property of addition, and others. In this case, the use of letter symbols helped the children to formulate appropriate propositions in general form and to cite concrete examples to illustrate a general rule.

The system we have outlined of progressively training pupils to construct equations in order to solve problems in the primary grades, of progressively introducing letter symbols as the generalization of the arithmetical knowledge which the children have built up, needs, of course, to be seriously tested by experiment. The experiment begun in 1963 in 18 classes of six schools has been expanded and is continuing, as we said

above. It is also necessary to experiment with other possible approaches to solving the problems raised. However, the data from the preliminary experiments described above demonstrate that an earlier introduction of elements of algebra is completely workable and can be successfully developed in the future.

CHAPTER V

GEOMETRY IN THE PRIMARY GRADES[1]

Questions of subject matter, system, and methods of studying geometric material in the primary grades are still far from settled. Recently, interest in them has grown especially in connection with the need to raise the level of mathematics education in school, to strengthen the connection between instruction and life, and to introduce polytechnical education. In the light of these goals, and the current state of affairs no one can be satisfied with geometry preparation in the primary grades.

Let us consider, first of all, how the aims of including elements of geometry in the primary mathematics course are stated in the explanatory note to the present curriculum:

> The chief task of geometric work in the primary grades is to give children clear-cut visual images of the line segment, angles, the rectangle, the square, the cube, and the rectangular parallelepiped; to consider certain properties of figures; and to use this knowledge to arm the children with practical skills in measuring length, area, and volume [12:53-54].

The importance of instilling spatial conceptions is also stressed in stating the general aims of arithmetic instruction, and in considering the bases of methodology--the practical direction of work in geometry.

Thus, at present, the study of geometric material in the primary grades is aimed chiefly at practical goals, since consideration of the properties of figures and the formation of appropriate concepts help to arm the pupils with the practical abilities and skills necessary for solving practical problems in calculating area or volume.

Perhaps, in connection with this, the selection of geometric material is largely a matter of chance. In fact, neither the explanatory

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[1] A. F. Govorkova participated in writing this chapter.

note to the curriculum nor the basic methods handbooks in arithmetic make any attempts to justify the selection of the material considered in grades 1 through 4. One can only suppose that figures such as the rectangle and square are chosen because of the relative frequency with which they are encountered in practical problems requiring calculation of area, and the ease with which these problems can be solved. Apparently, the same criteria dictate the choice of the cube and rectangular parallelepiped for determining volumes.

The curriculum and methodology noticeably fail to consider carefully what geometric work in the lower grades might best prepare the children for the regular geometry course later. The same tendency which we noted earlier in criticizing the insufficient use of generalization when working with primary schoolchildren, is clearly evident both in the selection of material and in the teaching methods.

Actually, the same procedure is followed here--teaching only certain practical material (mainly through sense perception) in the primary grades, and putting off generalization and the transition to operating with concepts until the future. Here the objective is not even to establish a link between the facts considered. Only this tendency to put off generalization can account for the stipulation in the curriculum that the children be familiar with the straight line and the line segment, but not with the half-line; that the children deal only with equilateral triangles during the entire four years of schooling, even though they are familiar with the triangle from the first steps of instruction, and study angles after that; that they are not even led to understand that the rectangle and the square are varieties of the quadrangle; and so on.

But even this direction in the curriculum is not followed with the necessary consistency. Suppose that the selection of figures is made according to the approach stated above. Then the question naturally comes up, why are figures such as the circle, and solids such as the sphere and the cylinder, familiar to the children and often encountered in their school and play activity and in practice generally, omitted from the curriculum?

Furthermore, if the development of children's spatial concepts is considered one important purpose of teaching geometric material in the

primary grades, then why not introduce exercises which would require the children to recognize familiar forms, not only in isolation, but also under more complex conditions, in which, say, a familiar figure is an element of a more complex configuration? This is clearly very important both for applying acquired knowledge to practical problems and for preparing for later work in geometry in the upper grades.

Furthermore, the authors of the curriculum and of textbooks are very cautious about using mathematical terminology. In order not to introduce the term "rectangular parallelepiped," they go so far as to state the rule for computing volume in this way:

> To calculate the volume of an object such as a drawer, a box, a room, etc., one must measure its length, width, and height, using the same unit of measure, and multiply the numbers obtained. The product is always in cubic units [50: 139].

We see that rejection of a term leads to an inaccurate statement. In other cases, although the circumlocution chosen to replace a mathematical term may describe the concept accurately, it takes longer to say. For example, "the sum of all the sides of the rectangle" instead of "the perimeter of the rectangle," and the like. Still, familiarity with the most common mathematical terms is necessary for successful continuation of instruction in the upper grades. The considerable difficulty children meet when they begin to study the regular geometry course occurs partly because in primary school they were poorly trained to perceive and use the terse, accurate mathematical language with which the mathematics teacher operates.

Observations and special experiments lately conducted by psychologists and methodologists give us reason to assert that the exceeding caution displayed in this respect in current teaching practice can be attributed only to an underestimate of the pupils' potential. In the light of the new goals confronting our schools, the necessity of revising the subject matter and methods of studying geometric material in the primary grades is obvious.

It is in this connection that in the practice of individual teachers, and in experiments by different groups of teachers, methodologists, and

scientific research institute workers, a wide scale search is being made for the most rational way of improving teaching in the primary grades in this direction. More and more often someone calls for expanding the content of the geometric material considered in the lower grades and proposes the introduction of a number of new figures (for examples, the parallelogram, rhombus, cylinder, and others) to supplement those now in the curriculum.

It is proposed to include in the curriculum new topics which are especially important for a more thorough and conscientious study of the properties of the figures considered (for example, parallel lines and the concepts of horizontal and vertical, among others). However, revision of the curriculum should consist not merely in amending or supplementing the nomenclature of the objects under consideration, or in adding a few new topics to those now included. It is important to eliminate the lack of system in the early introduction of geometry, and the lack of direction in working with this material.

In particular, it is necessary to decide what place geometry is to occupy in the elementary mathematics course. Should it be put to the service of arithmetic and function chiefly in the visual interpretation of arithmetical facts, laws, etc., being considered? Should geometry problems merely demonstrate one practical application of arithmetic, or should geometry be given independent importance in the course?

An analysis of the current curriculum has convinced us that as long as geometry occupies an insignificant, subordinate position in relation to arithmetic, it cannot serve to prepare children for the regular geometry course.

Obviously, the approach toward geometry in the mathematical training of younger students should be somewhat independent. But does this mean that geometry should be singled out as a separate, independent academic subject and that the study of arithmetic and geometry should proceed along parallel lines?

An idea that has recently gained wide prevalence, that of creating a single mathematics course combining arithmetic with elements of algebra and geometry, calls for a course in which all these elements are shown to be inseparably related. Of great interest in this connection is the question of what general approach might subsume the study of arithmetic and geometry in the primary grades.

Deserving of the most serious attention, we think, is the approach to solving this problem outlined in the works of Pyshkalo [58] and Neshkov [46], co-workers at the mathematics sector of the Institute of General and Polytechnical Education of the RSFSR, Academy of Pedagogical Sciences. The Institute developed just such a program for teaching first- through eighth-grade mathematics, in which arithmetic, algebra, and geometry are interrelated. The main idea unifying this course, and helping to show the organic connection between arithmetic and geometry problems, is the theoretical-quantitative approach.

Indeed, the operations of joining two quantities or of removing the appropriate part from a quantity, the concept of intersecting quantities, and the like, can all be interpreted both by using numerical quantities and by considering geometric figures as loci. Of course, this idea needs careful elaboration with regard to the goals of elementary instruction, and to the way primary schoolchildren think, and then experimental verification, but the work in this direction is interesting and productive.

However, the above approach, relating arithmetic to geometric material in the elementary mathematics course, cannot by itself determine what specific geometric material should be studied in the primary grades. The task of interrelating children's concrete and abstract thinking should play a primary role in solving this problem. Elementary instruction in geometry should by no means be limited to instilling certain spatial concepts and practical knowledge and skills in children. The procedure for teaching geometry should create conditions for more extensive use of generalization. It is necessary to plan the transition from notions to concepts. This need is dictated primarily by the aim of preparing children to study the regular geometry course.

However, in emphasizing the preparation of the pupils for later study of geometry, it should not be forgotten that the theoretical level of their geometry preparation can be raised only by systematically and purposefully enriching and developing their sensory experience, and that practical actions with concrete objects should help the children to discover the specific characteristics of their forms, and the relationships between different geometric figures and their elements, and thus help them to make suitable generalizations and abstractions.

For example, by solving the elementary practical problem of fashioning a triangle from three sticks of arbitrary length, the children will be convinced, from their own experience, that a triangle can be made not only from equal sticks but also from sticks of different lengths. And they will meet a case in which a triangle cannot be made from the sticks (the teacher should make sure this happens). Even if it is not stipulated that the children know how the sides of a triangle are related ("Every side of a triangle is less than the sum of the other two sides"), practical work in the primary grades will provide most valuable material for acquiring the appropriate knowledge later.

Practical work should be of help not only to accumulating material for later generalization, but also for teaching children more rational methods of operation. Thus, in solving different kinds of practical assignments related to measurement (for example, measuring a broken line), the children can actually perform various operations leading to the same goal by different paths, and from their own experience they can discern which of these ways is more rational. For example, in measuring a piece of wire which is bent in several places, they can measure the length of each section, and then add up the numbers obtained to determine the length of the piece; but they can also measure its length all at once, by straightening out the wire.

In organizing practical work for the pupils, one should always remember that concrete and abstract thinking should be developed interdependently. When ensuring that the children's sensory experience is broadened, one should simultaneously help them to develop the ability to abstract themselves from these immediate perceptions, which reflect certain irrelevant features of the figures as well as relevant ones. Inability to abstract oneself from an inessential can often hinder the formation of suitable concepts. Below we will dwell especially on this question, since we think it is of grave importance for inculcating a geometric approach to analyzing the objects in the environment.

In selecting the geometric material, it is also important to take into consideration what information about geometry the children desire or need in school and play and the daily conditions of life. In particular, it is necessary to give special attention to interdisciplinary

conditions. It is well known that in lessons in many academic subjects, particularly labor instruction, drawing, and natural history, students will need much of the information and many of the skills and habits that, by their nature, should be introduced in mathematics lessons.

At the same time, the connection between subjects should be established in other ways besides including, in the mathematics program, certain questions needed in work with other subjects. Obviously the connection should be two-sided. Thus, let us say, in work lessons or drawing lessons, there are extensive possibilities for clarifying and developing children's spatial notions and helping them to form "geometric vision"--the ability to see the form of objects, to compare the shapes of different objects in the environment. Work lessons should be used, in particular, for making various figures and for applying sketching and measuring skills to the job.

All this doubtless requires serious changes in the methodology of the teacher's work. Our goal is not to work out in any detail methods of considering geometry in primary school. We would only like to outline and discuss certain specific ways of improving present teaching practice which can be utilized even under the current curriculum.

First, if we want to raise the level of generalizations made in instruction in geometry (mentioned earlier in regard to arithmetical material), it is necessary to make a number of changes. For example, there is no reason why children who are familiar with triangles and quadrangles should be artificially protected from knowing other polygons. Even if, after explaining to the children why one figure is called a triangle (three angles) and another is called a quadrangle (four angles), we then showed them a pentagon, hexagon, and heptagon and told them to name these figures,* we would not say that this broadened the program. This would not be an examination of new figures but merely awareness of the principle of how they are designated. At the same time, this exercise would help the children to realize that the figures examined are particular cases and that a number of others may exist besides them.

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*Russian-speaking children could, because the suffix -ugol'nik remains the same for all of these terms (Trans.).

This is of no little importance for developing the children's thinking and extending their mathematical field of vision.

Furthermore, if we want them to form a general notion of, say, a triangle, we must by no means limit their experience to knowledge of the equilateral triangle, as is done now (this, by the way, is stipulated nowhere in the curriculum), but on the contrary, we should take care that children have the opportunity not only to observe but also constantly to meet triangles of various shapes in practical work. Sketching exercises, construction exercises, and the like, should be selected accordingly.

In Chapter I it was shown that the two most important conditions for forming concepts are wide variation of nonessential features, while keeping unchanged the essential features that enter into the substance of the concept being formed, and variation (complication) of the conditions under which the pupil is to apply the concept being formed.

These propositions can, in full measure, be extended to instilling generalized geometric concepts. Here it is important to vary the models the children are shown, and to change the conditions under which they are to recognize a certain figure. By slight modification of textbook assignments, one may use them to this end to a greater extent. In studying, say, the number 8, besides having the children make 2 small squares with 8 sticks, it is also useful to tell them to make a big square with the same sticks.

In addition to the figures given in the appendix of the first-grade text, one should use sets of cardboard squares and circles of diverse sizes and colors. Then it will be possible for the teacher repeatedly to call the children's attention to the fact that squares can differ in various ways, in color, size, and so on.

Not only should the size and color of the figures for demonstration vary, but also, without fail, their position in the plane (children often will not recognize familiar figures if a drawing shows them in an unusual position). Therefore, when setting the figures out on type-setting linen or attaching them to a board with plasticine, it is necessary to make sure that their position on the plane is not the same every time.

If it is necessary to vary the models of a certain shape widely in order to form general concepts, and then to differentiate the concepts, the most important thing is to compare various figures somewhat more often.

Exercises in figure-making can be very useful material for such comparisons. Thus, after the children have been told to make a square and a rectangle from sticks, it is useful to ask how many and what kind of sticks are needed to make a square, how many and what kind for a rectangle. By answering this question, the children become more aware of the difference between a square and a rectangle.

Exercises related to measurement, aside from their importance for evolving measuring skills, can also, under certain conditions, help the children to differentiate appropriate concepts. For example, before the children are told to measure the sides of a given square and a given rectangle, they can be asked which of the two figures depicted is the square and which is the rectangle. After the children name the figures and measure their sides, it is important to call their attention to the results of the measurement. If the teacher repeatedly calls their attention to the fact that two pairs of identical numbers are obtained every time the sides of a rectangle are measured, and that all four numbers are identical when the sides of a square are measured, it will serve as good preparation for considering the properties of the rectangle and the square in later grades and will be a basis for distinguishing these figures.

Of course, one cannot carry out the tasks of refining, generalizing, and differentiating the children's geometric concepts separately. These processes are all intimately related. However, first one and then another of these tasks may take precedence. It is important for the teacher to keep these tasks in front of him and to work conscientiously and purposefully to develop the children's spatial ideas and not hope that they will develop by themselves.

Later, the children should be taught to discriminate figures, to relate them to one category or another not merely on the basis of external similarity or difference, but on the basis of analysis---of sides (equal or unequal, what sides are equal) and angles (right or not). That is, by consciously singling out the significant features and abstracting them from the insignificant. The methodology should envision systematic work in this direction.

The process of mastering concepts is the process of transforming

what knowledge the children have into still more generalized and differentiated knowledge. Let us demonstrate this by example. Try this experiment with the children. Tell them to classify several small cardboard figures. Let the rectangles differ in the ratio of adjacent sides (6 cm. long and 3 cm. wide, 8 cm. long and 1 cm. wide, etc.). Tell the children to locate all the squares and all the rectangles, and to lay aside the figures that can be put neither with the squares nor with the rectangles. The children will find the squares, but they will name as rectangles only those figures whose width is two or three times less than their length. Long, narrow rectangles they will call "rulers, "little stripes," or "columns." This mistake is quite often encountered in children. To what can it be ascribed? The point is that the changing ratio of the sides changes the external appearance of the figures, making them bear little resemblance to one another. Relying on the external resemblance of the figures, the children do not regard figures dissimilar to the standard ones as rectangles.

It is very important for the child to have similar difficulty later when he learns other concepts of geometry (parallelogram, trapezoid, etc.). There the changing ratio of the sides makes variations of the same geometric shape dissimilar. The mistakes will disappear only when the children learn to relate a number of figures to one category (say, rectangles), not on the basis of superficial similarity but by discerning essential features and abstracting them from the inessentials. Thus, knowledge of the properties of figures and mastery of the concepts "square" and "rectangle" allows us to overcome the limitation on knowledge that inevitably results from letting the children approach geometric figures from the standpoint of superficial resemblances and differences. As children form geometric concepts, their knowledge of shapes rises to a new level and becomes more generalized and differentiated.

However, it is not enough simply to impart to the pupils information about the properties of geometric figures; it is necessary to organize work for mastering these properties. We shall not list here exercises which can be used to this end. We shall only observe that those in which it is necessary to use knowledge of the properties of figures actively, to manipulate geometric forms, should occupy a central position among them.

In exercises which require the children to measure the sides of a given rectangle, to find the perimeter of a square with one side equal to so many centimeters, or to draw a rectangle whose sides are known, the children deal with an assigned and named figure. But to develop concepts of shapes, assignments which require them to determine what the figure is, how it differs from another figure, why it can be called a rectangle, and so on, are of great importance. The text has very few assignments of this type. However, the teacher can easily supplement the textbook by having the children work independently with teaching materials. For example, every pupil gets a small card depicting a square and a rectangle that is nearly a square in the ratio of its sides. The assignment is stated like this: "Measure the sides of the figures on the card and determine which one is the square."

In doing the assignments discussed above, it is more often necessary to use one of the essential features of a concept (equality of sides: all the sides of a square, opposite sides of a rectangle). It is also important, however, to give the children exercises which will bring another feature, right angles, into consideration.

The role of this second feature can be demonstrated to them only if exercises are conducted in recognizing right-angled figures among figures that are not right-angled, as well as by comparing a square and rectangle. Especially valuable in this regard are assignments requiring the children to recognize squares among rhombuses, rectangles among parallelograms.

The curriculum for the primary grades does not stipulate the introduction of rhombuses and parallelograms. Therefore in conducting the corresponding exercises, one may distinguish between a "square" and a "non-square," a "rectangle" and a "non-rectangle." In practice, appropriate classwork can be conducted in the following manner. The teacher distributes cards to the children which show geometric figures (two or three, and a larger number later) in various positions: for example, let one card show a square and rectangle, another--a square and a rhombus, a third--a square, a rhombus, and a rectangle (or rectangle and parallelogram), and the like. The children are to find a square among the given figures and to measure its side (if the work is conducted in second grade), or to calculate the sum of all the sides of the square (third grade), or calculate the area of a rectangle (fourth grade), and the like.

Such exercises will make the children actively use the knowledge they have acquired and will facilitate the formation of the necessary generalizations.

If, further, we wish to develop propositions whose comprehension requires intelligent abstract thinking by the children, which will not remain empty phrases for them, then we should plan a system of exercises in which these abstractions will be replete with concrete substance. For example, if we introduce the concept that a straight line can be continued as far in both directions as one likes, then it is necessary to reinforce this proposition by making the children apply it in practice. Pyshkalo and Neshkov used a good exercise of this kind in their experiment [58]: The children were given a drawing of a straight line and several points on the line, outside of it, and on the mental continuation of the line segment traced on the picture. The pupils had to enumerate the points lying on the given line.

We have already mentioned that the ability to see and recognize familiar figures, not just when they are isolated in a drawing but also when they are elements of more complex configurations, has great significance for later geometry work in the upper grades. If we want the children actually to learn this, we must create teaching conditions under which they will repeatedly carry out the proper exercises.

The simplest of them is, let us say, to enumerate all the segments obtained when on a straight line there are given four points, all the half-lines that can be formed here, and so on. When introducing a figure as a locus of points, great possibilities are opened up for considering various instances of the intersection of two or of several figures. Suitable exercises can easily be related to an explanation of the new figures obtained.

Quite useful, too, are exercises in which the children themselves must construct new figures from several given ones--for example, constructing a rectangle or a square from given triangles (to make the suitable materials, it is enough to cut up a rectangle or square into the appropriate parts). The children receive both these exercises, and those which call for superimposing, rotation, parallel shifting of figures and the like, with great interest [71].

Elaborating and systematizing such exercises, and searching for more successful methods of conducting them, is one of the most important tasks now confronting the methodology of elementary mathematics teaching.

It is perfectly clear that all the complex problems which arise in connection with creating a single elementary mathematics course, an organic part of which should be geometric material, can be solved only on the basis of special experiments. But even now in teaching practice, teachers can do much to improve the preparation of primary-grade pupils for the regular geometry course.

The accumulation of experience in setting up systematic exercises intended to form and develop concrete notions of shape, spatial position, and object size, exercises whose purpose is to instill the necessary generalizations and the first geometric concepts, can render an inestimable service in subsequently developing the methodology of elementary mathematics instruction.

CHAPTER VI

DIFFERENT KINDS OF PUPILS AND HOW TO APPROACH THEM IN ARITHMETIC INSTRUCTION

The preceding sections discussed ways to improve methods of teaching arithmetic, guided by knowledge of the way in which schoolchildren assimilate material. Knowledge of the general features of learning is very important for making methods of teaching arithmetic more effective, but it is not enough. In the same class, at the same level of instruction, one can encounter children who differ greatly. Some comprehend the material easily, while others are passive, and so on.

Teaching can achieve good results only when the individual psychological differences disclosed in the process are taken into account. But the question naturally arises, where and how are these differences revealed? Their manifestations are extremely varied, and the educationally suitable method for approaching a child can be ensured only if we know what individual traits displayed in the process of instruction are typical or characteristic of the pupils as a group.

This chapter is devoted to a consideration of the features typical of the way primary schoolchildren learn arithmetic. In revealing these peculiarities, we shall be describing an individualized approach to children in the teaching procedure. It should be stressed that such an approach can be effective only if it is consistently implemented at all stages of instruction: when new material is introduced, when independent exercises are organized, when the pupils recite, and in their homework assignments.

There is still another kind of work made especially for slow pupils--supplementary assignments. These sometimes do not bring about the desired results, but only because they are used with the very limited objective of giving the pupils additional practice with something they have not mastered sufficiently. This narrow objective is attainable by supplementary exercises, but when these pupils go on to new material, they

have similar difficulties and again lag behind.

To overcome this and thereby forestall failure, it is necessary, in effecting an individual approach at all stages of instruction, to set higher goals, that is, to change the individual psychological traits of the pupils by cultivating those qualities of mind and character which are insufficiently developed, arousing their interest in arithmetic, and forming a creative, initiatory approach to school activity, using every opportunity to develop their capabilities.

However, accomplishment of these broader tasks, under the conventional classroom system of instruction, encounters a number of difficulties. Under these conditions, possibilities for individualizing the teaching procedure are very limited. At the same time, avoidance of failure and of repetition of a year depends to a substantial degree upon expanding the opportunities for individualizing instruction. But how can organized class activity be changed to this end?

Teachers are working on this question. Deserving of particular attention is the experiment of Budarnyi, who heads the Methodological Center of Kalinin District of Moscow and who, together with teachers, is putting new forms of lesson organization into effect [8].

By this system, the class is divided into three groups, depending on typical learning traits, or as the authors expresses it, on their "scholastic aptitudes"--high, average, or low. The pupil groups work with three different versions of the assignments, gaining knowledge, ability, and skills at their level. With such an organization, optimal conditions are created for the pupils' systematic growth, for transfer from one group to another--from a group of poorer pupils to a group of better ones.

The educational aspect has considerable importance in this system. The children who learn the material poorly have the opportunity of getting high marks by receiving assignments within their powers, and this has a positive influence on their attitude toward schoolwork and awakens a desire to learn even better.

Such an organization of the lesson is being implemented experimentally in the primary grades, group work being combined with work of the class as a whole. Further and expanded verification of this experiment is needed, serious work on creating differentiated assignments is in prospect,

and many methodological problems are still unresolved. But one thing is already clear; it is necessary to conduct the lesson in such a way that the distinctive learning traits of different students are taken into consideration.

In this chapter, we shall pay particular attention to the question of the form in which children manifest certain traits when they are assimilating various arithmetic materials at different years of schooling. Thus the reader, having learned from this chapter how certain individual psychological peculiarities of schoolchildren are manifested, will be able to draw conclusions in two directions: first, how to study the children by bringing out their distinctive traits, and second, the proper ways of approaching each pupil individually.

We will give basic attention to describing two types of pupils: those with a high aptitude for arithmetic and those who learn arithmetic with difficulty. We have chosen to describe two contrasting types of pupils, but we will not limit ourselves to showing the psychological features only in these, the most striking manifestations, but will try to show how deviations from the type of pupil described might be displayed.

Thus, we direct the teacher's attention to discernment not only of similar but of different traits when he studies the pupils, and we point out that sometimes different psychological traits hide behind outwardly similar ones. Relying on such a study, the teacher will be able, in working with the class, to outline methods of individual treatment which should be applied to the entire group of pupils, and along with this, to plan work with individual pupils, taking their uniqueness into consideration.

In every section of this book we discuss the most effective ways of approaching pupils individually. However, this question requires further elaboration, with the active participation of teachers.

1. Pupils with a high aptitude for arithmetic

This type of pupil usually reveals himself very quickly in the first days of school. These pupils, as a rule, come to school with a better fund of knowledge; in particular, they come prepared for school instruction, preparation which shows up in their ability to subordinate their activity to the tasks set by the teacher. They easily master the rules

of conduct in school and comply with the demands made on them in the learning process.

The most striking feature of their academic activity is the quick tempo at which they learn school material. Quickness in learning is the external sign behind which lie hidden definite qualitative features of the learning process and a higher level of mental activity, which is implemented in learning.

This level is characterized, primarily, by more highly developed analysis and synthesis, generalization, abstraction, and concretization. This can easily be discovered as instruction progresses and with various school materials. Perhaps a child who enters school in this category does not know a great many numerals and knows how to count by units only up to 10. This is not important. The essential thing is something else--how the pupil will determine quantity in the number figures offered him, such as :: :: . The child who performs analysis and synthesis at a higher level will immediately notice and select a convenient grouping of dots and use it in counting. Instead of counting the dots one at a time, he will resort to the more rational method of converting the number figure into a cardinal number and count by groups ("4 and 4 make 8").[1]

The pupil who shows an aptitude for arithmetic will make a generalization by himself in cases when many of his classmates still do not have one at their disposal. For example, on the basis of his experience with performing the operation of addition (with objects), he will quickly come to the conclusion that the numerical result does not change when the name of the objects being calculated is changed. Naturally, he still will not state it in the form of a rule, but he will use this generalization in practice--when he receives a particular numerical result from computing with certain objects (for example, "8 matches"), he will carry it over to other objects (blocks, pencils, etc.), saying immediately, without recounting, "That's 8, too." This single word "too" expresses generalization and abstraction, since the pupil distinguishes an essential, general

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[1]Naturally we mean here children who have been taught in kindergarten to count by groups using numerical figures. As psychological research shows, under these identical conditions, some children master more complete forms of analysis and synthesis when they operate with numbers, while others only count by units.

property (the correlation of numerical operation and result) and divorces himself from nonessential features (the qualitative aspect of the calculated objects).

To ascertain to what degree a first grader knows how to discern essential features, we can offer him a problem in which inessential elements of the condition are deliberately stressed, while certain essentials are, so to speak, put at a disadvantage. Such, for example, is this arithmetic problem:

> A boy went to the woods for mushrooms 2 times. He found 5 white mushrooms the first time and three white mushrooms the second time. How many white mushrooms in all did the boy find?

It is not hard to see that an insignificant datum ("2 times"), having no bearing on the arithmetical content of the problem, is written as a digit here (thereby prompting the children to use it), while a significant element of the condition ("three mushrooms"), on the contrary, is expressed not as a digit but in letters, something the children are not used to. Pupils with high aptitude in arithmetic solve this problem. They discard the extra datum and use the necessary numerical datum correctly even though it is written in letters.

Pupils in this category accurately and subtly delimit concepts and systems of knowledge and skills. A number of systems of knowledge and skills which are introduced in the mathematics course have similar features. The two ways of designating numbers when solving problems in "division into parts" and "division by content," which children have to learn in second grade, are a conspicuous example of these systems.

Many mistakes occur with this kind of problem. Either one system of designation is substituted for the other, or isolated elements of the two systems are confused. Pupils who learn arithmetic easily manage even this difficult task. As they become familiar with a condition, they are aware of it in general, that is, even in the initial analysis they relate the problem to a particular type. From the very start, this makes them aim at a definite form of writing the numerical data when solving problems. In the one case (division into parts) they write a concrete dividend, an abstract divisor, and a quotient with the same concrete name as the dividend. In the second case (division by content) they

also write a concrete dividend, but then change the form for writing the concrete names, in comparison with the first case, by putting down a divisor of the same name as the dividend. They write the quotient with another name, in brackets, or they leave it as an abstract number.

In solving problems of this kind, the solution should be checked thoroughly, and it is necessary to keep a generalized description of the type of problem being solved constantly in mind. This feature of perceiving academic material more generally, of immediately relating it to a definite category of phenomena (on the basis of rules or principles studied), is a very important trait of pupils who learn arithmetic easily.

They have this kind of generalized perception, or more accurately, recognition, in regard to type-problems. In this connection, recognizing the type of problem studied, they are guided by the sum total of significant features, not just by any individual features that happen to strike the eye. They manage to perceive the relation between data and unknown behind a reworded condition. They separate the familiar part of the condition from the new, extra part and distinguish (differentiate) problems of similar types.

Such pupils can easily make problems more concrete. The class advances to operations with abstract numbers, such as doing table multiplication "in their heads," but as soon as they are told, these pupils can return to previous stages and make this operation more concrete. That is, they can demonstrate it with objects as the addition of equal addends.

In this connection, a very important trait shows up--a flexible or mobile mind, by virtue of which the pupil can, on his own initiative, change (vary) calculation techniques. For example, if he forgets a result in the multiplication table, he either reproduces the result of the nearest combination of numbers he can remember from the table and, with this as a guide, determines the desired product, or he replaces multiplication by the addition of equal addends.

Knowing how to use various techniques of calculating, the children at the same time know how to limit themselves in using them, selecting the most rational of them, depending on the features of the example being solved. The pupils' discovery of characteristic features of assigned examples hinges on an ability to analyze.

Schoolwork continually confronts children with the task of changing the method of operation according to a change in the condition, and they must often solve a series of similar assignments, after which comes another one that requires them to change their method of operation. In such cases our pupils notice the change of condition and reorganize their method of operation accordingly. For example, after solving a series of addition examples, at the right time they notice a change in sign in the next example and carry out subtraction correctly.

Besides assignments of this kind, pupils in the upper grades have to solve more complex ones, overcoming the analogous tendency to repeat the preceding operation. This occurs in subsuming a series of particular cases under a general rule. Here it is not enough to discern any one element (for example, the operation sign); rather, the relation among all the elements of the condition must be established. For example, when third graders apply the commutative law of addition to solve the examples,

$$7 + 3 + 2 + 7 = 19,$$
$$3 + 2 + 7 + 7 = ?$$

they do not make calculations in the second line, because they notice that it has the same numbers as the first line. And although they have had to apply this law over and over again, some of them, those with a flexible mind, do not make a mistake even when, right after these examples, they are given an example outwardly similar but not subject to the given rule:

$$6 + 2 + 4 = 12,$$
$$2 + 5 + 6 = ?$$

So in this case, overcoming the desire to act by analogy, they notice the new addend in the second line, though to do so they have to carry out the rather complex operation of comparing two rows of digits.

Children learning arithmetic are often required to reinterpret certain principles studied earlier. And pupils with flexible minds manage to do it successfully, although it gives great difficulty to the others. For example, in over two years of practice in solving examples the children become accustomed to the necessity of performing operations with numbers in the order in which they are written. Then in the third year they meet a new rule: The solving order may coincide with the written order, but it also may not coincide, depending on the operations (higher or lower)

one must deal with.

This task is especially complicated when the children have to apply both these principles one right after the other. To prove this to yourself, give third- and fourth-grade pupils these three examples after they study operation order:

1) $8 \times 3 + 6 =$
2) $5 + 6 \times 4 =$
3) $2 \times 3 + 4 \times 5 =$

In the first one, the order of solving coincides with the order in which the numbers are written; in the second and third examples it does not. A number of pupils will make mistakes when solving the last two examples, likening them to the first and performing the operations with the numbers in the order in which they were written.

Students with flexible minds, however, can cope with this assignment. They use not only the first, familiar rule in solving the examples but also the second--the new one--although it contradicts their usual experience. It is characteristic of these pupils that studying a new rule reorganizes the very way they perceive the operations of arithmetic, as operations of different stages. This is seen from how they substantiate the order in which they carried out the operation--they do not state a rule but only indicate what stage the operation belongs to.

Flexibility of mind can manifest itself in the most varied forms, not only in solving examples and problems (as discussed above), but also in composing problems. In this kind of activity, the ability of the children to vary the problem's story to express a single operation of arithmetic is of special interest. For example, directions such as these are given: "Think up a problem where you have to add." After the pupil has composed a problem, he is again told to think up a problem including the same operation.

Let us quote a set of addition problems devised by one of the pupils who learn arithmetic easily.

> Problem 1: "Masha had 3 kilograms of acorns; the second time they went to the woods she gathered 6 kilograms more. How many kilograms does she have altogether?
>
> Problem 2: "First a boy picked 6 apples from an apple tree, then 4 more. How many apples did he pick?"

Problem 3: "Three cars were driving down the road, then 6 more cars drove by. How many cars were driving down the road?"

We see that the story in the problem changes every time, and the everyday operation that expresses addition ("gathered," "picked," "drove by") varies accordingly. We see that the boy is not confined to a single formulation.

We have been trying to show in what form certain traits are manifested in children who learn arithmetic easily. Activeness and flexibility of mind are manifested most strikingly in that these children approach a new story problem as a mystery whose methods of solving must be found; they do not rush into calculation but analyze the problem's condition. As they proceed to solve partial problems, they keep the condition of the problem as a whole within the sphere of their attention; as they repeatedly ask themselves whether a given operation needs to be performed, they constantly keep in mind the problem's last question and the other data in the condition. The most difficult kind of analysis--analysis directed toward subsequent operations and characterized by "anticipating"--is within the powers of these pupils when they are still in the primary grades (of course, with the proper teaching methods).

It is also very important for them to have doubts while they are solving ("I don't think it'll work this way--I'll see what happens if I divide," etc.). Having transformed the problem's condition, they continue the analysis, reject methods they have used, if necessary, and again search for the right ones.

Some begin applying graphic analysis (if the teacher suggests it), that is, they illustrate in diagram form the relation, expressed in the problem's condition, between data and unknown.

All of this characterizes the creative, or as they still say, the "productive" approach to problem solving. It is closely allied with an interest in difficult problems, with the awakening of the so-called "intellectual" feelings, and above all, with enjoyment of mental activity itself, of successfully solving a new problem without assistance.

Of course, these pupils also make mistakes sometimes (especially in solving more difficult problems by themselves), but being emotionally involved in their mistakes, they quickly overcome them.

Typical of the category of pupils we have described are a painstaking and responsible attitude towards study and a high receptivity to the moral aims cultivated by the school. For these children, a conscientious attitude toward their obligations as pupils is natural and necessary.

Pupils in this category display organization and concentration in their studies. If we observe them preparing their lessons, we will notice that they begin by preparing the work space. They carefully familiarize themselves with the assignment; if they read the text aloud, they observe the proper intonations. Some of them, in doing an assignment (an arithmetic problem, for example), whisper or reason out aloud in order to facilitate their own understanding. Where the work causes no problems they do not use speech. They work with concentration and are not distracted as they carry out assignments in different subjects.

We have described the type of pupils with the most highly developed positive qualities. Their activity and flexibility of mind combine with an emotionally colored, positive attitude toward study, with interest, a sense of duty as a pupil, and organization of their work.

This category of children should be approached, above all, by systematically providing them with enough material to reinforce and further develop their positive qualities during instruction. In working with the class, there is always the danger of losing sight of these pupils while giving special attention to pulling up the laggers. Making those learn who are distinguished by their slowness, can hinder the development of children of higher aptitude. Without sufficient sustenance to satisfy their interests and intellectual activity, they will gradually lose their positive qualities--their interest will be spent, they will show passivity. They may even become discipline problems. In what specific ways should children with a high aptitude for academic material be approached as a group? With this in view, individualization of assignments for independent work is necessary--along with the examples and

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[2] A comprehensive description of primary schoolchildren, particularly of this, the most numerous, group with a responsible attitude toward study is given in [69].

problems solved by the whole class, the teacher should always have an additional number of assignments of increased difficulty to give the children who cope with the classwork more quickly. To this end, Popova's Didactic Material may be used with benefit in all four years of instruction.

The teacher has extensive opportunities to utilize the pupils' intellectual activity when he introduces new material, by letting the pupils participate directly in deriving a general rule or law by noticing common properties in numerical facts, helping to put the rule or law into words and inventing new, particular examples supporting it. The pupils can take an active part in checking papers by other pupils, and in various forms of activity out of class, aimed at developing their mathematical abilities. The mathematical olympiads organized for primary schoolchildren are valuable experience.

We have described the most striking type of pupils, who have a whole aggregate of positive traits. But it is perfectly natural that in life, in school practice, we should encounter the most diverse representatives of this type of child, closer to or farther from this "ideal" we have described, who also have certain weak points: For example, the pupil who is capable of high-level intellectual activity, but in whom an insufficiently responsible attitude toward study engenders a certain unsteadiness in his work—he works actively only to satisfy an immediate interest, or he does not have habits of orgarized work. There may be substantial deviation in the characteristic features of a pupil's thinking: Quickness in generalizing may combine with a certain slowness in switching from one method of operation to another, in the processes of generalized recognition; the pupil may reveal an inclination to be guided by inessential, external criteria, and so on.

That is why it is very important for the teacher, in systematically studying the children in the course of schoolwork, to expose their weaker sides so that a harmonious development of all their positive qualities may be achieved.

2. Pupils with a lower aptitude for arithmetic

In this section we shall maintain the same order of exposition.

After giving a description of the psychological features of this type of pupil, we shall go on to discuss the most effective ways of approaching this group of children. Then we shall touch upon possible divergencies from this type of child and the changes that must be made in an individual approach.

Frequently, difficulties in assimilating material are also combined with difficulties in learning rules of conduct. In this connection, the pupil, so to speak, "walks away" from the task set by the teacher, submitting to his demands neither in academic nor in instructive work. This shows up in the first days of school.

For example, in the first days of arithmetic instruction in first grade, the children are told to sketch a certain number of objects in their notebooks, corresponding to the number and digit they are studying. But some children display characteristic mistakes. The number of objects in the drawing does not correspond to the given number and digit. They sometimes draw as many as the notebook can hold. Thus the children, carried away by the process of drawing, overlook the academic task. A similar phenomenon may occur at this stage of instruction when the children are given the assignment of expanding a number. For example, they are told to spread out six circles however they want onto the left and right sides of the notebook. A substantial part of the class uses a definite principle of expansion, in accordance with the assignment, while individual pupils, overlooking the assigned task, draw circles in any order, not bothering to correlate the rows in both columns, and they feel that the task is completed when the whole sheet of paper is crammed with circles.

Subsequent observation of these children during schoolwork discloses that they are notable for the slow tempo at which they absorb the material. Most pupils in the class have already gone on to more complete calculation by groups of units in connection with addition, while these children continue to employ the most elementary method--counting by 1's. Or, when a significant part of the class has gone on to abstract counting, these children are still counting on their fingers, with sticks, or the like.

This slow learning tempo testifies to a low level of mental activity. Weakness in generalizing shows up most vividly. A pupil will repeatedly

add 3 and 4 by using sticks, for example, but when asked how much do 4 pencils and 3 pencils make, he either will answer "I don't know" or will get some pencils to count them.

An inclination to rely on purely external, inessential criteria shows up very clearly in the generalizations of such a child. Thus in solving arithmetic problems, to him the main and decisive thing is not the meaning of the problem but rather certain formal aspects, as, for example, that the number is written as a digit. Give such a pupil a problem in which one of the data is written in letters, and this will create difficulties for him.

> Nadya had 4 kittens. She gave one of them to a friend. How many kittens did Nadya have left?

After reading the problem through, the pupil immediately names the first numerical datum, "Add 4 and...," but here he pauses, saying in amazement, "And what?" He didn't find the second numerical datum because it was not written as a numeral. Here we see not only weakness at generalization, but another feature inseparably linked with low mental development--inertness of thinking, its low mobility.

This group of pupils staunchly clings to customary methods of operating and tends to repeat an operation over and over again. This is manifested in various degrees in different pupils. Let us describe the most extreme manifestations of this peculiarity.

When examples are solved, one of the numbers in the condition is written (repeated) in the result: $8 - 7 + 9 = 9$ (instead of 10) or $2 + 6 - 3 = 2$ (instead of 5), and the like. When a series of examples are solved one after the other, the preceeding example is fixed in the mind, thereby excluding the new problem. This exclusion may have to do with the operation sign, so that the pupil continues to carry out the operation he did before without noticing a change of sign, or it concerns the nature of the numbers. For example, a first grader would initially be solving examples in adding a two-digit number and a one-digit number; the next column has examples in adding one-digit numbers; he solves them incorrectly, likening a one-digit number to a two-digit number, under the influence of the examples done earlier.

When independently composing problems, the most distinctive feature of the slow pupil's procedure is the extreme uniformity of the story aspect of the problem and, in particular, of the words that determine the operation. For example, take these three addition problems that one first-grade pupil thought up:

1) A little girl first bought 5 meters, then 2 more meters. How many were there in all?

2) A little girl had 10 books, and she bought 10 more books. How many were there in all?

3) There were 10 flowers in the window, and mother bought 10 more flowers. How many were there in all?

It is easy to see that the same word, "bought," and the same question, "How many were there in all?" figure in all three problems. Thus the pupil makes the task much easier for himself by not essentially varying the wording of the condition.

We have cited a series of examples, showing how, in carrying out one study assignment (or a series of assignments one after the other), the pupil obstinately keeps the preceding operation in his mind, refusing to admit the next new problem. A similar phenomenon takes place when the study of different topics is separated by a period of time.

The more familiar sections of the material, which were studied previously, influence the study of later ones in such a way that newer methods of operation are replaced by ones the children are more accustomed to.

Highly indicative in this respect are the mistakes first graders make when they go from solving and constructing examples to constructing story problems. For example, when told to construct an addition problem, a pupil says: "At first there were 4 airplanes flying, then 3 more.... 5, 6, 7, 8," With this he concludes the problem without having formulated a question, and including in the condition a (wrong) numerical result obtained by counting. In reply to the request made right afterwards, to think up a subtraction problem ("where you have to take away"), he does not respond to the changed instructions but again, ignoring the question of the problem, includes a numerical result in the condition: "First 10

airplanes were flying, and 3 more flew after them; 3 more were added to the 10....10, 11, 12, 13, then there were 13."

Thus, due to extremely inert thinking, the pupil reproduces the operation he is most used to--counting—likening the structure of a problem to the more familiar structure of an example.

Slow pupils often show a tendency to replace mental calculation with written calculations "in their heads," after they study methods of written calculation in third grade. For example, one pupil, in mentally multiplying 75 by 3, calculates this way: "5 × 3 = 15, I write 5 and remember 1; 3 × 7 = 21 and the 1 in my head makes it 22, I get 225." Curiously, the pupils prefer this wholly uneconomical method (one must visualize the digits as written) only because they can then rely on more definite rules which have become for them, so to speak, "stereotyped" rules of calculation, and can thereby avoid mental calculation, which permits a variation of calculation methods and requires them to choose the most expedient ones.

Pupils are observed to replace new methods of operating with the old, more habitual ones when they solve arithmetic problems, as reflected above all in the way they write the names of the numerical data. Every teacher can detect this easily by observing his pupils. They have begun to solve multiplication problems and in this case one of the data--the multiplier--should be written without names, while in solving addition and subtraction problems it was always necessary to write the names. Watch how the pupils write names when they go on to solving the multiplication problems. There will undoubtedly be pupils in the class who at this stage make the mistake of writing, "3 notebooks × 3 notebooks = 9 notebooks"--i.e., they will also name the multiplier.

We happened to observe this phenomenon: A teacher worked especially with the little boy who made this kind of mistake and the boy began to write names properly for the operation of multiplication. But when he had to solve an addition problem, he got it wrong, having written the operation of addition according to the multiplication model: "12 kopeks + 2 = 14 kopeks."

It is quite clear that in this case supplementary exercises in a certain operation are not enough. A whole system of exercises calculated to form more flexible thinking, to overcome the trait of inertness, is

needed. If special educational work is not conducted with such pupils, they pass from grade to grade, retaining these negative features of thinking, and because of these peculiarities, they subsequently strive to avoid any relatively new task, relying only on the procedure of mechanically reproducing the operations immediately preceding or the operations which they have often encountered in their earlier academic experience.

The first section of this chapter cited a number of assignments for pupils in the third and fourth grades which make it possible to discover exactly how they react to a new problem, and whether they can change their method of operation if necessary. Let us briefly recall one of them, having shown how it is solved by pupils with low aptitude for arithmetic. Third graders are given examples for applying the commutative law of addition. These consist of two rows of addends. In the first row the result is written down, and in the other one it has to be determined by the pupil. He is asked, "Can you tell right away, without figuring, what the answer will be when the numbers in the lower row are added?" In this connection, along with examples permitting application of the law, examples are purposely introduced to which it cannot be applied.

It is found that the pupils who did not have a flexible mind, after having applied the commutative law to the first three examples, incorrectly continue to apply it in the fourth and sixth, to which it no longer applies; thus, they simply repeat the previous method of operation without analyzing the addends in both rows, without comparing them with each other, and therefore do not notice that the addends in these rows do not fully coincide.

The urge to repeat previous operations <u>without analyzing the assignment itself</u> is a characteristic feature of this category of pupils. Some may manifest this trait quite persistently, even right after they have been shown their mistake, thus demonstrating the necessity of analyzing the assignment. For example, one of the third graders under study erroneously used the commutative law on the fourth test example. When she was asked to check, she noticed her mistake: "The numbers aren't the same here." However, going on to the next one--the fifth example (which permitted application of the law)--she incorrectly rejected the opportunity of using it in this case: "The numbers aren't the same

here, either--you can't." She simply reproduced the previous operation, again refusing to analyze the assignment.

The desire to avoid analysis of the assignment is usually accompanied in these pupils by lack of checking. That is, they exhibit a lack of analysis of their own solving procedure and of the result obtained. It is manifested in all forms of their activity--both in solving examples and in solving problems. Thus when doing written calculations, the pupil manipulates figures without even thinking about what numbers he gets as a result, and he does not correlate them with the example's condition.

We had occasion to observe one third-grade girl who, in dividing a three-digit number by a one-digit number, obtained as the quotient a number exceeding the dividend in size, and yet did not notice the absurdity of her numerical result. Let us cite only one example of such a calculation, which was not isolated. She divided 748 by 4 in the following manner:

```
   997
4 )748
  - 36
    38
  - 36
     28
  - 28
     00
```

She detached not one but two digits in the dividend (not having noticed that 7 is divisible by 4); dividing 74 by 4, she got the quotient's first digit, 9. She divided the remainder by 4 again, getting 2 as a remainder, brought down the last digit of the dividend and, having divided 28 by 4, thought that she had solved the example quite correctly. She did not even notice that the quotient (997) came out larger than the dividend (748) since she made no comparisons—the numbers with which she operated actually lacked any concrete meaning for her. It should be noted that in this girl's class, her classmates knew how to determine the number of digits in the quotient beforehand, since their teacher had taught them this especially (see A. A. Tsank's article in the collection cited in [3]).

Analogously, when solving problems, the pupils in this group solve a problem as long as they manage to do any calculations, solving separate, familiar problems included within the composite problem. And they are

observed to pause only when some calculation cannot be performed ("It couldn't be divided," for example). But in the process of solving, they do not stop to ask themselves whether they have chosen the correct operation, whether certain numerical data could be correlated, whether it was necessary from the standpoint of the problem's question. In other words, they do not check the process of their mental activity, so that their psychological work does not coincide with the problem they are told to solve. In their effort to avoid any difficulty or novelty, they continually, in the course of studying, falsely substitute problems, transforming a problem useful in a cognitive respect into an assignment requiring no mental effort at all.

This inclination to avoid any active mental activity is manifested still more strikingly in the solution of type-problems. Let us analyze how slow pupils learn methods of solving type-problems. To reveal this we gave individual instruction to two fourth-grade girls who solved problems poorly. A problem in equating data was chosen. The problem is no longer part of the program, but for our experiments it made no difference, since it was necessary to choose a type of problem that had not yet been solved in class. Many repetitions were required (17 for one girl, 19 for the other) before they learned to solve similar problems of this type (while for the other pupils in the class, two, three, or at most nine repetitions were enough).

Typically, mistakes in choosing an arithmetical operation were the first to disappear (with mistakes in formulating a question disappearing later); i.e., the order of arithmetical operations in solving a problem was learned first of all.

It was also easy to see that the operation was not linked to the question as a whole, which expresses a particular idea, but to one of its elements which in the pupils' previous experience was united with a definite operation most of the time: "how many more--subtraction" "how many times as much--multiplication," and the like.

Thus, the children had worked out very short chains of links ("associations") between external, nonessential, features of the condition and a definite arithmetical operation. These links, applied to a problem of equating data, could take this form: "The last number in the condition and the smaller number of the first two numbers must be multiplied."

"The larger and the smaller number (of the first pair of numerical data) must be subtracted," and the like.

It is not surprising that these pupils needed so many repetitions to learn a type-method. Indeed, it is known that memorizing meaningless material is more difficult than remembering material that has geen given meaning and can be expanded into meaningful parts. Pupils like those described show a tremendous waste of labor, but they achieve nothing, even after learning how to solve analogous problems of a definite type, since this is not a "solution of the problem" in the real sense of the word.

This assertion is easy to check by telling them to solve problems of this type but with certain changes in the condition. You change the condition's wording, and slow pupils, it appears, can deal successfully with the new assignment. Instead of:

> 14 small furnaces and 4 large furnaces smelt 4500 tons of steel a day at one plant. A large and small furnace smelt 750 tons of steel. How much does a large one smelt?

a problem like this is given:

> A student bought 3 books and 5 notebooks and paid 40 kopeks the first time. The next time she bought only 1 book and 1 notebook at the same prices and paid 12 kopeks for this purchase. How much do a book and a notebook cost separately.

But this is only seeming prosperity. The pupils managed this assignment only because they did not scrutinize the meaning of the problem, and the external aspects of the condition--the number, quantity, and arrangement of the numerical data--permitted the very same operation sequence (multiplication, twofold subtraction, division, and then subtraction again). In solving this problem, the better pupils typically had difficulties, for they made an effort to investigate the meaning of the problem and could not immediately ascertain that this change of wording did not alter the mathematical nature of the problem.

But now you bring another nonessential change into the problem, changing the number of numerical data by introducing an extra condition:

A housewife bought 2 kilograms of potatoes and 4 kilograms of cabbage and paid 20 kopeks in 4 coins for the entire purchase. What are the prices of potatoes and cabbage if 1 kilogram of potatoes and 1 kilogram of cabbage together cost 25 kopeks?

Your below-average pupils will display utter confusion because the arithmetical operation sequence they have learned by heart has been violated. They will take the number denoting weight from the number denoting cost, and so on.

Now try making a relevant change of condition. Give them another type of problem (a "substitution" problem) that has individual features of resemblance to the original problem in equating data. The pupils in this case will act as if no changes had been introduced. In this problem, the total cost of one item from each of two categories is not given, but rather, how many times more expensive one item is than another is known. But the pupils handle this numerical datum as if it designated the total cost. They multiply it by the smaller of the first pair of numbers. For example, solving the problem,

Twelve lemons and 20 oranges cost 6 rubles, 24 kopeks. An orange is twice as expensive as a lemon. How much does a lemon cost and how much does an orange cost?

they multiply the number "2" by "12," and ask the question, "How much do 12 lemons and 12 oranges cost?"

We have been describing the problem-solving procedure of those pupils who completely lack the ability to solve problems. Cases of this kind are not often encountered in school even among pupils who have trouble learning arithmetic. Partial inability to solve problems is observed significantly more often, in which weakness of analysis is manifested. If a versatile analysis of the condition is characteristic of above-average pupils (as was noted above), so that they do not lose sight of the problem as a whole in solving each partial problem, and, in choosing each operation, are aware not only of the problem's question but also of later data, then the opposite is characteristic of below-average children. Analysis is performed in short steps; it is always directed at some one element without taking into account everything else, without "anticipating."

This peculiarity is particularly obvious when compound problems are solved in which it is possible to make a "false synthesis," when the numerical data can be correlated in various ways: both correctly (when the problem as a whole is taken into account) and incorrectly. For example, in solving the problem,

> A store sold 84 kilograms of cake the first day and 192 kilograms the second day. 86 rubles 40 kopeks more was received for them on the second day than on the first day. In the 2 days, how much money was taken in for the cake altogether?

the first two numerical data (84 kilograms and 192 kilograms) may incorrectly be added if there is no "anticipating," that is, if analysis directed at the subsequent data and at the question of the problem is lacking.

The pupils who do not know how to solve problems perform this false synthesis because, having selected this pair of data from the condition, they perform the first possible synthesis that comes into their heads, paying no attention to the next datum, "86 rubles 40 kopeks more was received the second day than on the first day," and it is just this datum that requires them not to add the amount of cake sold during the 2 days, but to ascertain the difference in the quantity of goods sold.

The frequency with which certain numerical data have been correlated in practice is of no small importance for the way below-average pupils solve. From this standpoint, problems in which it is necessary to perform a synthesis which sharply contradicts the synthesis to which the pupils are accustomed are of particular interest. Let us given an example of such a problem:

> Three boxes of cargo, 170 kilograms in all, were loaded onto a truck. The first box weighed 60 kilograms, and the second box weighed 8 kilograms more than the third. Determine the amount of cargo in the second and third boxes.

The problem is not hard, but if you give it to a fourth-grade class, you will find several pupils who get it wrong. And this mistake is of a definite kind: They will add 60 kilograms and 8 kilograms as if the condition had said, "the second box weighed 8 kilograms more than the first." This can be explained by the frequent use of such relationships in previous practice.

Some children very persistently manifest such a fixed idea about habitual combinations of data, and it strongly hinders solution of the problem. Thus, one of the fourth-grade pupils we studied made a mistake in reading the condition of the problem cited above: "more than the second," he read incorrectly. The mistake was corrected immediately by the teacher, but even so, the boy incorrectly added 60 kilograms and 8 kilograms when he did the problem. So even in reading the problem the pupil introduces the content to which he is accustomed.

All of the cases described are phenomena of a single order. Underdevelopment of the mental processes engenders great difficulties in learning the material, and in reaction to these difficulties, a special approach to learning is elaborated which may be called "reproductive," since it involves the mechanical reproduction of the study material. In this case, the pupil strives to adapt the study assignments to his limited capabilities. He systematically avoids active thinking everywhere. And this inevitably brings his development to a complete halt.

To complete this description of the reproductive approach to learning, we must address our attention to one more peculiarity of pupils in this category, which we have only touched upon until now. Operation with empty stock phrases which have no concrete impressions behind them, no perceptions or conceptions of real objects or of actions with these objects, is characteristic of the school activity of these children. Below-average pupils (as opposed to those with a high aptitude for arithmetic), who have passed with difficulty from calculating with concrete objects to calculating "in their heads," afterwards cannot show their process of calculating with real objects. Just as abstraction had given them trouble earlier, now the reverse process--concretization--gives them trouble. In solving problems they try, above all, to depend on individual words (and word combinations) without trying clearly to imagine what specific images are behind these words, what real circumstances are described in the problem's condition.

Usually this peculiarity is called formalism or verbalism in learning. Sometimes these terms designate a superficial, shallow acquisition of knowledge, which may be engendered in many pupils in the same class by faulty teaching methods.

But we have something else in mind--a trait inherent in individual children which is characterized by a definite, relatively static approach to learning. Such formalism is inseparably linked with a pupil's inclination to rely on mechanical rote-learning and reproduction of the purely external, at times inessential aspects of academic material in learning something. It also assumes as one of its most characteristic features the pupil's inability to see the abstract, the general, in the specific, the particular, and the reverse, an inability to perceive the specific and particular in the abstract and general. Pupils in this category (as opposed to those characterized by their high degree of assimilation) do not perceive a school assignment in a generalized way, relating it to a definite group of phenomena. And if, for example, they do relate a problem to a certain type, it is done not by grasping the sum total of the relevant characteristics of the given type, but by taking account of the first inessential feature that strikes the eye (if the condition says "how many times as large," the problem is division "into parts").

This feature of formal or verbal learning, however, may have different origins in different pupils. This has to be investigated. Before us are two pupils, characterized by the manifest verbalism of their knowledge. Outwardly their learning processes are quite similar, but from a psychological standpoint there is a substantial difference between these pupils. Both strive to replace mental calculation by written comparison, trying to use it even when they are calculating "in their heads," and both in fact "walk away" from problem solving, being guided in the process of this imaginary "solving" by purely formal word rules: "When we say 'remaining' we have to subtract," "It says 'together' here, so you have to add," and the like. When given the problem, "There are enough oats for 8 horses for 6 days. For how many days is there enough for one horse?" one fourth grader, having read the condition, quickly started to solve. "We can find out....," but suddenly he stopped and said after a pause: "It doesn't come out." When asked what stopped him from solving the problem, he explained, "You can't divide 6 by 8." Thus, without having tried to understand the specific content of the problem, he reproduced a customary datum reflecting formal, external (and irrelevant) relationships. "If they ask about one thing, you have to divide."

But what is the difference between these two pupils? Clarifying this question required further study. It was necessary to go beyond the bounds of academic activity and to determine how their perceptions proceed, what their conceptions are, how developed their movements are; in other words, one must establish the degree to which their sensory-motor processes are developed and the extent of their sense experience. It was found that they differed in this particular respect. One (a third-grade girl, to whom learning came with extreme difficulty) did not have elementary notions of everyday things; she lacked the simplest skills that require the use of motor, muscular, and tactile sensations. For example, she could not compare weights of objects identical in size, she could not cut a square into two or four equal parts, she was not capable of recognizing from among objects shown her those which she had seen immediately before--that is, her processes of visual recognition were not developed.

By contrast, the other pupil (who was making very poor progress, and whom we studied for three years, from first through third grade) was a totally normal child in the development of the sensory-motor processes went. He even coped quite successfully with a more difficult assignment in visual recognition.

Consequently, these two pupils lagged behind for two different reasons: in one case, there was underdevelopment of the sensory processes, which explains the excessive verbalism of knowledge, the use of empty words--words lacking concrete meaning--while in the other case, unjustified verbalism of knowledge with a normally developed sensory basis meant only that the necessary connections had not been formed in due course between the words and their sensory images.

Cases in the first category are extremely rare in the environment of the mass school. But the teacher needs to recognize them. In school we more often encounter cases in the second category, for whom there is more hope, but even with them very serious work is needed, since usually such a formal attitude towards words (and therefore, towards schoolwork) is reinforced over a number of years of schooling, and then a gap between knowledge and life opens up for these pupils, so that the learning process becomes more difficult for them every passing year.

All pupils in the category of those who learn with difficulty and

resort to methods of formal verbal learning are not, as a rule, fond of mental effort. The intellectual feelings are foreign to them--the joy of discovery, of surmounting difficulties in solving problems, the sense of satisfaction which comes from strenuous mental activity. They often shirk difficulties in schoolwork: "I don't know how," "I can't" is the answer frequently heard from them. They are not sufficiently concerned about their mistakes, and therefore surmount them with difficulty. Many of them answer without thinking, making mistake after mistake, and arrive at the right answer through a series of wrong ones.

For example, one third-grade girl added 8 and 32 and got 52 as the answer. To the question, "What do you think, is the answer right or not?" she replied without checking that it was correct. When the mistake was pointed out, she quickly changed her answer to 51. When told a second time, she answered 50, and learning that this was wrong, she again named 52 as the result. Only after the error of her result was pointed out for the fourth time did she find the correct answer of 40. The most striking thing about the way she answered was her utter lack of interest in getting the right answer, the indifference to the results she attained.

In the pupils who strikingly display negative qualities of personality which hinder study, lower learning capacity is combined with an irresponsible attitude toward study and low receptivity to the moral aims being implanted by the school. Often they have a desire to study in the beginning, but it disappears as a result of systematic failure to study, and is replaced by a formal and devil-may-care attitude toward academic activity. Usually more complaints are made about the carelessness of these children than anything else. And indeed, their easy distraction by any external stimulus is characteristic of them both in class and in doing independent work at home. But this carelessness is usually not the primary cause of their study problems; it is itself a result of the very many causes discussed above: unreceptivity to moral aims, a passive personality, underdeveloped intellectual processes.

Disorganization and inability to work also play a very large role. Observe these pupils as they do their homework and you will see how they work: They do not prepare the work space ahead of time and so are constantly distracted from their studies by their search for a book, notebook, or pencil. They do their lessons in random order; without finishing

one thing, they proceed to another and do not check what they have done. Pay particular attention to how they read a problem. In many cases you may discover that it is read with incorrect intonations which completely distort the meaning of the problem. For example, one slow third grader read this arithmetic problem [69: 125-131]:

> A sovkhoz turned in 3020 tons of wheat and rye to the state. 965 tons of oats less than wheat.
>
> Five times less than the wheat and rye. Together. How many tons of grain in all did the sovkhoz turn in to the state?*

It is typical that this boy gave most of his attention to pronouncing individual words distinctly, but the connection between the words which determines the problem's meaning totally escaped his notice. Naturally, in reading like that, he could not understand the problem's condition. When asked how he understood the problem, he was silent for a long time, and then he answered, "I don't know. I can't solve the problem."

From this answer it can be seen that he did not even try to pierce the meaning of the text; he set for himself a completely different "psychological task"--to read the individual words smoothly and distinctly. And only after the psychologist working with him instructed him in how to read a problem, breaking down the text of it into meaningful segments, did the content of the problem and the entire solution become comprehensible to the pupil.

By observing how second through fourth graders read a problem, we can detect other instances. A pupil hurriedly reads the condition of an arithmetic problem just as he would any other descriptive writing, even though a special type of reading is needed here: i.e., isolating the data from the question, stressing those word combinations which express a relationship between data and unknown. Reading it once is not enough. The first reading need not give attention to the data in figures, but in the second reading the figure data are the object of attention. Thus, reading an arithmetic problem's condition is, to a certain degree, an analysis of the problem. And inability to analyze may show up in the way the pupil reads a problem.

So we have characterized the most striking representatives of the group of pupils distinguished by their low aptitude for learning academic

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* Clarification of this problem can be found in the Appendix. (Ed.)

material. They combine passive and inert thinking with avoidance of any mental effort, an indifferent or even negative attitude toward study, and carelessness and disorganization in working. There are not many of these pupils in any class, but success in preventing them from failing and repeating a grade depends on exposing them in time and on organizing special educational work with them.

Basically, these pupils are the ones who are failing. Eventually they become repeaters, unless special, systematic work is conducted with them.

It should be remarked, above all, that the general shortcomings in teaching methods show up most negatively in the slow children. If, for example, the teacher does not vary the concrete material sufficiently when he explains concepts and laws, it is the slow children who assume inessential features to be the basis of generalization, while quick pupils add to what the teacher did not give in class themselves, from their own experience, on the strength of their active intellects. Or, similarly, if the teacher does not employ contrast in introducing similar concepts, poor pupils will confuse these concepts, while the strong pupils will catch the distinguishing features and delimit one from another without any contrast, without the teacher's help.

Completeness of the general methodology of teaching, its conformity to scientific requirements, is very important in preventing pupils from lagging, but this is still not enough. Supplementary work with individual pupils is necessary. If there are several in the class, then it is necessary to work with the whole group. Supplementary classes with the group are needed for this, chiefly, not when it has become clear that they are behind the class in learning certain sections of the curriculum (as is usually done), but much earlier, before new material is introduced in class. In these lessons, one should give supplementary exercises which the group of children need in order to master the new material. Then when this material is introduced for the first time in class, our slow pupils will be in a more favorable position and will be able to participate in the work together with their classmates.

But in working with slow pupils, one cannot rely only on special supplementary lessons with them (even if such lessons protect them from

possible difficulties). A special approach to these pupils during sessions with the whole class, actively drawing them into group work, is of decisive significance.

Sometimes a teacher prefers not to question slow pupils when he is explaining new material, fearing that they will hold back the rest of the class. But at this stage one should enlist their active cooperation; moreover, it is possible to prepare them for this well beforehand. It is useful to give grades for answers on new material which has just been explained by the teacher. K. A. Moskalenko proves the expediency of this in a special paper [45]. The teachers of the Lipetskaya Region found this method very effective in practice, which undoubtedly is linked to the general system of making studies more dynamic which was put into effect by these teachers. Apparently, the psychological advantage of this method is that it makes the pupil an active participant at all stages of the lesson. Furthermore, a grade provides the check that the teacher needs on how well the pupils understand new material. The latter is especially important for increasing the effectiveness of work with poor pupils.

From this standpoint, all the kinds of independent work organized in class by the teacher have still greater potential. The exercise material should be individualized. Popova's Didactic Material, which is used in the schools, makes such an individualization of the assignments easier for the teacher. The third-grade teaching material compiled by Polyak is issued especially with this aim. Assignments for different groups of pupils have been prepared by teachers of Moscow schools according to Budarnyi's system: A. S. Lazareva (School No. 844), T. A. Zhuravleva (School No. 784), and M. F. Ivelva (Boarding School No. 44).

Thus, both general teaching methods and supplementary lessons with poor pupils, as well as an individual approach in classroom procedure, can facilitate both a change in their personality traits and in the qualities of mind which hinder them from studying, and the formation of higher learning capabilities.

For this, any assignment, for individual work as well as for work with a group of pupils or with the whole class, should pursue several aims at once: to nourish their intellectual activity, to awaken the desire for intellectual effort, to form a positive attitude toward study, a love for and an interest in independent work, to instill habits of

organization, and to train them to concentrate longer.

Both the exercise material and the classroom method must serve these ends. The most important principle in selecting material is moving the focus of attention from solution to analysis of the assignment. Problems should often be given, not to be solved, but only to determine what operation has to be performed and with what numerical data. In this connection, the children need to be asked why a certain choice was made. Exercises in determining problem type are also valuable, and in this connection, one should require the pupils to explain why the problem was put under this type, i.e., they should be required to indicate the criteria according to which the problem was related to a definite type.

Depending on how the pupils master the operation of analysis, one need not require them to substantiate the operation being performed every time, but one should give them exercises of various kinds which promotes formation of this skill: problems with missing number data, problems with extra elements, problems which have sufficient number data but can be solved, without using numbers, merely by analyzing the condition, among others (examples of such problems are given in Chapter I).

Along with the operations of analysis and synthesis, it is necessary to make the childrer practice abstraction and concretization, giving special exercises for this purpose: to deduce a law or rule from a series of particular examples illustrating it, to write down the steps of the solution as an algebraic formula, to relate a specific problem to a definite type, and conversely, to give their own examples illustrating some law or rule, to recreate the concrete condition of the problem on the basis of the algebraic formula, to make up a series of problems of a definite type, and so forth.

It is also very important to teach the children how to translate a numerical operation done "in the head" into a scheme of concrete operations with objects, and if necessary, to "deautomatize" a numerical operation (that is, to make the automatic operation a conscious one), when doing written calculations, so that the pupil will become aware of the principles of notation and of what figures represent, what numbers they express.

Teaching how to switch from one method of operation to another demands special attention. To this end it is necessary to make assignments of various degrees of difficulty by assigning examples in different operations alternately, and to increase the difficulty by repeating the same operation in a great number of examples and then giving an example in another operation.[3]

Exercises in applying any rule, or exercises in solving problems, can be constructed analogously. Type-problems provide especially good material for these exercises--there it is possible to alternate similar types of problems, as well as to alternate type-problems with non-type-problems which have certain elements of resemblance to a type-problem.[4]

A necessary condition for working with poor pupils effectively is the teacher's constant check on the progress of their mental process, their methods of solving. In bringing to light the way they perform when solving examples and problems, it is necessary systematically to teach them more rational methods of working independently (preparing the work space, doing the assignment in a definite sequence--from the harder to the easier, checking what has been done, using the textbook expediently, and the like). Special attention should be given to teaching methods of solving problems independently (discussed in detail in Chapter IV).

The most important thing in working with poor pupils is to change their approach to a problem, replacing their tendency to reproduce habitual methods of solving by the desire to search for methods of solving a problem which is relatively new to them. Speaking briefly and using special terminology, we might say that the "reproductive objective" should be replaced by a "productive" (that is, creative) one. It is necessary to demonstrate visually to these children (where possible) the gross mistakes to which their unwillingness to think and their inability to notice what is new in the condition of a problem may lead them. To

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[3] On the question of "switching" in arithmetic, see [33].

[4] The principles of varying problems have been discussed in preceding chapters.

the same end, one can sometimes give them special problems having a mathematical absurdity in the condition, and use their mistakes to stress more sharply that it is necessary to understand the meaning of the problem every time. One should drill them especially in distinguishing similar problems and in bringing to light how their conditions are distinct from those of problems solved earlier.

But even observance of all these conditions will not lead to the objective if the general-education aspect is overlooked: the inculcation of a positive attitude toward study and an interest in performing intellectual tasks. It is fitting to approach this goal from various aspects at once: on the one hand, one must relate what the child is learning to his experience, to his life, and on the other hand, one must reinforce his faith in his own abilities by encouraging him wherever possible. One can always create an excuse for encouraging a poor pupil by giving him easier assignments which can be assumed to be completely within his grasp. It is necessary to remember one psychological phenomenon: bad grades and reproaches continually addressed to the same pupil ultimately blunt his sensitivity to his own mistakes and engender indifference, and eventually an irresponsible attitude toward his studies. Censure can make psychological sense and be effective only when it is alternated with encouragement, and used in a strictly differentiated way, that is, only when it is really necessary.

Sometimes it is educationally permissible, in evaluating a single answer (oral or written), both to reproach and encourage, separating, for example, the quality of the work (which is still unsatisfactory) from the attempt to perform it without assistance, applying certain more complete methods, and making an effort, all of which deserve a positive evaluation ("good try!" and the like).

Of great significance, especially for poor pupils, is the teacher's employment not only of a quantitative grade (in numbers*) but also a qualitative grade based on analysis both of the result itself (the final answer) and of the methods used by the pupils. Many teachers justly give attention to the importance of this point, effecting this approach in their own practice.

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*Russian schools give number grades, 5 being the top mark (Trans.).

We have described typical features of pupils with a low aptitude for arithmetic by showing these features in their most striking manifestations, in which the difficulties of learning are combined with a negative attitude toward study, lack of organization, etc. Later, we described an individual approach to this category of pupils. However, when using this approach in teaching practice upon children who are having difficulties in arithmetic, one must keep in mind the extreme diversity of their psychological traits and the possibility of the widest divergence from the type we have described above.

It should be taken into consideration that every pupil has his strong points, which also must eventually be discerned in order that one may be guided by them in working with him. One pupil may be characterized by a very low learning capacity, but his diligence and desire to study can help to compensate for this shortcoming and undoubtedly will play a large role in developing the qualities he lacks. The learning capacity of another pupil, on the other hand, is not significantly low, but his education is greatly impeded as a result of his disorganization, lack of discipline, and stubborn reluctance to study. Naturally, in approaching him, one needs to use his broader mental capabilities and, guided by his growing learning capacity, to excite his interest in academic activity, at the same time reinforcing his discipline and forming habits of organization.

In getting to know a child, one should always try to single out some leading feature, on whose development primarily depends the development of the child's personality and his success in schoolwork. Sometimes children are encountered who are extremely shy, easily hurt, and who have trouble in social intercourse with their classmates. With such a child, the most important thing a person can do is to "uninhibit" him, to accustom him to the group, after overcoming his shyness and reinforcing his faith in himself and his abilities. Here one must begin with educational work. If this is not conducted, no later systematic training in solving mental problems can be effective. This training should be effected after, then along with, realization of the educational aims of altering the pupil's character (in this respect, Fevraleva's experiment, which she describes in detail, is instructive [20]).

Also very important in analyzing a pupil's mental activity is to expose that peculiarity of his which decisively influences the entire style of his schoolwork, and is one of the fundamental causes of his difficulties in his studies. Let us cite some examples.

We observed two pupils, first in the fourth year of instruction, and then in fifth grade. Both experienced substantial difficulties in arithmetic instruction, but a careful study (made with the help of the teacher) revealed essential differences in the character of these difficulties and their causes.

The basic shortcoming which hindered one of the pupils from studying was lack of originality, inertness of thought. When studying, this boy would always try to avoid every difficulty, every necessity of making mental effort; therefore he took the course of reproducing verbal knowledge mechanically, without correlating it with specific impressions, with reality. This boy was of the type described above. It was he who, in solving the problem,

> There are enough oats for 8 horses for 6 days. For how many days is there enough for one horse?

was ready to perform division, without trying to understand how the real quantities were related by proceeding from a word formula ("If they ask about one thing, you have to divide"). When he was asked "How does the difference vary when the subtrahend is increased?" he answered, "It gets bigger," again not trying to understand the specific meaning of the concepts ("difference," "subtrahend") and following the most familiar train of thought, "If they increase it, it gets bigger." Also characteristic of this child was the constant influence of preceding operations, and his extreme difficulty in switching from one method of operation to another. For example, having solved one example in operation order, in which no operation took precedence, he analogously solved a second example in which an operation took precedence and in which it was necessary to perform operations in a different order from the one in which they were written.

His classmate also had difficulties in arithmetic, but he tried to overcome them--on his own he tried to concretize the abstract concepts he had to deal with, but this slowed down the process of his work.

Solving the problem cited above by a simple triple rule with inversely proportioned quantities, he was confused, not by the fact that one number was not divisible by another (as his friend was), but by not knowing "how much oats one horse eats." This difficulty testified to the fact that he had tried to understand the specific content of the problem. In answer to the question about how the difference changes when the subtrahend is increased, he used a concrete example which helped him to comprehend the necessary relationships: "If I have 5 rubles and I spend not just 2 rubles but 3 rubles, then I've got less left."

Since this pupil always tried to grasp the specific meaning of the assignment, he did not make mistakes under the conditions of having to switch from one procedure to another. This was his strong point. But he felt the need to make something specific even when the problem could easily be solved with abstract logical thinking. For example, in solving the problem, "There were 60 kilograms of grapes in two boxes. Sixteen kilograms were taken from the first box and 16 kilograms were put into the second. How many kilograms of grapes were there in both boxes then?" he again resorted to a concrete example ("If I've got 5 rubles and I spend 4, and they give me another 4 rubles, then I'll have 5 rubles") instead of interpreting this change in a generalized way: "They added as many as they took away" [41].

Discovering the psychological features of learning in these two pupils makes it possible to plan various ways of approaching them individually in the teaching process. It is necessary in the first case continually to train the pupil to relate abstract concepts to reality, to perceptions and conceptions of this reality, while in the second case something else is needed--to strengthen abstract logical thinking, gradually weaning the pupil from dependence upon concrete methods in uncovering an abstract relation. But the way to develop thinking in the second case is simpler, since in this child there exists the necessary concrete basis for thinking, while it still must be created in his classmate, who suffers from excessive verbalism of knowledge.

Thus, in working out an individual approach, it is necessary to be guided by a thorough study of the pupils, discovering both their typical (group) features and those features which create the individual uniqueness

of their psychological makeup.

One question of great, fundamental importance arises. What should be the goal of the teaching-process--should one erase or level the children's individual psychological differences, or, conversely, after exposing certain distinctive peculiarities of the pupils, should one try to develop them, thereby deepening the existing differences between students?

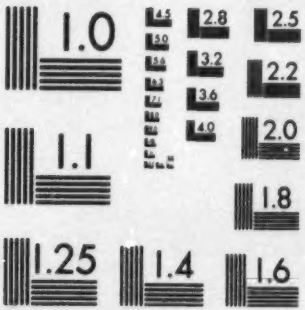
It would be wrong to answer this question categorically, accepting either one solution of the other. Both need to be done, but with regard to different personal qualities. The goal should be to develop in all children a positive attitude toward study and active, independent, versatile thinking, thereby leveling children's differences in these characteristics of personality. This should also be the task in forming academic and practical abilities and skills; again, they should be brought to a high level in all children.

These are the foremost tasks of the primary school, and that is why this chapter has chiefly considered them. But in the process of primary education we need to set ourselves another goal at the same time, which will be fully realized in later years, in the middle and upper grades. We mean the aim of strengthening or deepening individual psychological differences. This should involve primarily such aspects of the personality as inclinations and interests, which may be linked with a particular cast of mind and with clearly expressed aptitudes in scientific or artistic creativity.

During the years of primary education, a number of children may manifest an aptitude for technology or artistic creativity in its various forms. Some children show a very striking interest in nature, and the young naturalists become proficient in natural science, going beyond the curriculum.

And finally, in primary school young mathematicians can be found who appreciably distinguish themselves from the other children by the quickness and thoroughness with which they learn mathematical material and by their original ways of thinking in solving problems.

These differences--differences of inclination and aptitude-- must not be leveled; on the contrary, they should be developed in every way,



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thus facilitating the development of the personality in all the wealth and uniqueness of its individual manifestations.

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APPENDIX

The passage on page 184 quoted from Volokina's Essays in the Psychology of Primary Schoolchildren [69] is likely to be unclear to the reader who does not have access to the original. According to Volokina, the slow third grader in question had trouble with a particular homework problem although he usually performed calculations with ease. His native tongue was the Tatar' language, but he spoke and read Russian quite well. Occasionally he did confuse gender inflections (which exist in Russian but not in Tatar').

Although neither Menchinskaya and Moro nor Volokina gives the text of the problem as it appeared in the textbook, it must have looked as follows (with the original word order in Russian preserved so as to clarify the pupil's errors in reading):

A sovkhoz delivered to the state 3,020 tons of wheat; of rye--
965 tons less than of wheat; of oats--
five times less than of wheat and rye
together. How many tons of grain in all did the sovkhoz deliver to the state

Volokina observes that when presented with the problem, the pupil sat motionless, silently reading it over with an indifferent expression on his face, apparently understanding nothing. When asked by the psychologist what the problem was about, he answered, "About tons." Then he was asked, "What kind of tons are they generally talking about in the problem and what must you find out?" He looked aside and said nothing. To learn what was causing the boy difficulty, the psychologist asked him to read the problem aloud. The boy gave the following reading:

A sovkhoz delivered to the state 3,020 tons of wheat and of rye.
965 tons less than of wheat, of oats.
Five times less than of wheat and of rye.
Together. How many tons of grain in all did the sovkhoz deliver to the state

After instruction by the psychologist (as indicated on page 184), the boy finally wrote the problem in his exercise book. Volokina quotes what he wrote:

A sovkhoz delivered to the state:
3,020 tons of wheat,
of rye, 965 tons less than of wheat,
of oats, five times less than that of wheat and rye together.
How many tons of grain in all did the sovkhoz deliver to the state?

Since the boy was not a native Russian, he had some difficulty understanding the technical style in which problems are typically stated. Further, his lack of understanding of individual words and expressions appeared to make it difficult for him to grasp the sense of a problem already partitioned into its components. His incorrect reading of the problem resulted in arbitrary pauses that further obscured its meaning. He had developed an attitude of despair toward problem solving, figuring before he began that whatever he did would be useless. According to Volokina, however, once he was able to make the semantic connection between words, his morale rose and his performance improved. (Ed.)